Catch-up and Fall-back through Innovation and Imitation
Online Technical Appendix

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References with no prefix refer to this Technical Appendix, while those with a prefix of BPT. refer to equations in the main paper.

4 Concavity and sufficient parameter restrictions

Define the maximized Hamiltonian $\hat{H}(x,\lambda) \equiv \max_{\gamma,s} H(x,s,\gamma,\lambda)$, where the Hamiltonian, H, is

$$H(x,s,\gamma,\lambda) = \ln(x) + \lambda x\sigma\gamma + \ln(B-s-\gamma) + \lambda x\left(\tfrac{c}{m}(1-x^m)s - g\right).$$

Lemma 1. *The maximized Hamiltonian, $\hat{H}(x,\lambda)$, is concave in x if $m \geq -1$ and $\frac{c}{\sigma} < m+2$.*

Note that both conditions in the Lemma above restrict the efficiency of technology diffusion.

Proof. Differentiating $H(x,s,\gamma,\lambda)$ shows

$$H_{ss} = -\frac{1}{(B-s-\gamma)^2};\ \gamma = 0, \tag{1}$$
$$H_{sx} = \tfrac{c}{m}\lambda\left(1-(1+m)x^m\right);\ \gamma = 0, \tag{2}$$
$$H_{xx} = -\frac{1+c(1+m)\lambda s x^{1+m}}{x^2};\ \gamma = 0, \tag{3}$$
$$H_{\gamma\gamma} = -\frac{1}{(B-s-\gamma)^2};\ s = 0, \tag{4}$$
$$H_{\gamma x} = \sigma\lambda;\ s = 0, \tag{5}$$
$$H_{xx} = -x^{-2};\ s = 0. \tag{6}$$

Over the innovation-only region characterized by $x > x^*$, $\gamma(x,\lambda) > 0$, so we can compute $\frac{\partial\gamma}{\partial x}$ from the first order condition $H_\gamma = 0$. Then, by the envelope condition, $\frac{d\hat{H}}{dx} = H_x + H_\gamma\frac{\partial\gamma}{\partial s}$ and $\frac{d^2\hat{H}}{dx^2} = H_{xx} + H_{x\gamma}\frac{\partial\gamma}{\partial s}$, which must be non-positive to assert the concavity of $\hat{H}(x,\lambda)$ in x. Thus, the concavity of $\hat{H}(x,\lambda)$ in the innovation-only region follows if $\frac{d^2\hat{H}}{dx^2} = H_{xx} + H_{x\gamma}\frac{\partial\gamma}{\partial x} = H_{xx} - \frac{(H_{x\gamma})^2}{H_{\gamma\gamma}} \leq 0$, since $\gamma > 0$ and, thus, $H_\gamma = 0$. Therefore, in the region $\frac{c}{m\sigma}(1-x^m) < 1$ where $s = 0$ and $\gamma > 0$, we have $H_{\gamma\gamma} = -(B-\gamma)^{-2} < 0$, $H_{xx} = -x^{-2} < 0$, and $H_{\gamma x} = \sigma\lambda > 0$. Thus,

$$H_{xx} - \frac{(H_{\gamma x})^2}{H_{\gamma\gamma}} = -x^{-2} + (B-\gamma)^2(\sigma\lambda)^2 \tag{7}$$
$$= -x^{-2} + \frac{(\sigma\lambda)^2}{(z\sigma\lambda)^2} = 0. \tag{8}$$

This is enough to establish the required concavity over the innovation-only region $(x^*,1)$ in which $s=0$, and, together with the first order optimality conditions, establishes the optimal policy in the region where $\gamma(x,\lambda) > 0$ and $s = 0$.

Establishing concavity in the complimentary region of $x < x^*$, in which $s \geq 0$ and $\gamma = 0$, is more complicated. Since $\gamma = 0$ in this region, either $s = 0$ over the interior of a sub-interval where $\frac{\partial s}{\partial x} = 0$ or $s > 0$ so that $\frac{\partial s}{\partial x}$ can be computed from the first order condition $H_s = 0$. So, either $H_s = 0$ or $\frac{\partial s}{\partial x} = 0$. Then, by the envelope condition, $\frac{d\hat{H}}{dx} = H_x + H_s \frac{\partial s}{\partial x}$ and $\frac{d^2\hat{H}}{dx^2} = H_{xx} + H_{xs}\frac{\partial s}{\partial x}$, which must be non-positive to assert the concavity $\hat{H}(x, \lambda)$. The concavity of $\hat{H}(x, \lambda)$ in that region will then follow if $\frac{d^2\hat{H}}{dx^2} = H_{xx} + H_{xs}\frac{\partial s}{\partial x} = H_{xx} - \frac{(H_{xs})^2}{H_{ss}} < 0$ if $H_s = 0$ or, $\frac{d^2\hat{H}}{dx^2} = H_{xx}$ if s is in the interior of a sub-interval where $s = 0$. We have:

$$H_{ss} = -\frac{1}{(B-s)^2};\ \gamma = 0, \tag{9}$$

$$H_{sx} = \tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right);\ \gamma = 0, \tag{10}$$

$$H_{xx} = -\frac{1 + c(1+m)\lambda s x^{1+m}}{x^2};\ \gamma = 0. \tag{11}$$

Therefore we need $Q = H_{xx} - \frac{(H_{sx})^2}{H_{ss}} < 0$. Evaluating we obtain:

$$Q = -\frac{1 + c(1+m)\lambda s x^{1+m}}{x^2} + \left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 (B-s)^2. \tag{12}$$

The first order condition for s in the region $x < x^*$ is:

$$\frac{1}{B-s} \geq \lambda x \frac{c}{m}\left(1 - x^m\right), \tag{13}$$

$$B - s \leq \left(\lambda x \frac{c}{m}\left(1 - x^m\right)\right)^{-1}, \tag{14}$$

$$(B-s)^2 \leq \left(\lambda x \frac{c}{m}\left(1 - x^m\right)\right)^{-2}. \tag{15}$$

From Eq. (12), $\hat{H}(x, \lambda)$ is concave if $Q < 0$ and, since $x > 0$, $\hat{H}(x, \lambda)$ is concave if $Qx^2 < 0$. Using the FOC shows:

$$Qx^2 = -\left(1 + c(1+m)\lambda s x^{1+m}\right) + x^2\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 (B-s)^2 \tag{16}$$

$$\leq -\left(1 + c(1+m)\lambda s x^{1+m}\right) + x^2\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 \left(\lambda x \frac{c}{m}\left(1 - (x)^m\right)\right)^{-2}. \tag{17}$$

Then, $\hat{H}(x, \lambda)$ is concave in x if

$$-\left(1 + c(1+m)\lambda s x^{1+m}\right) + x^2\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 \left(\lambda x \frac{c}{m}\left(1 - (x)^m\right)\right)^{-2} < 0,$$

$$x^2\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 \left(\lambda x \frac{c}{m}\left(1 - x^m\right)\right)^{-2} < 1 + c(1+m)\lambda s x^{1+m}. \tag{18}$$

Since $1 + c(1+m)\lambda s x^{1+m} \geq 1$ for $m \geq -1$, we can obtain sufficient conditions that ensure $Q < 0$ by checking for conditions under which

$$x^2\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)^2 \left(\lambda x \frac{c}{m}\left(1 - x^m\right)\right)^{-2} < 1$$

in the region $x < x^*$. Reorganizing Eq. (18) shows that

$$\left(\frac{\left(\tfrac{c}{m}\lambda\left(1 - (1+m)x^m\right)\right)}{\left(\lambda\tfrac{c}{m}\left(1 - x^m\right)\right)}\right)^2 < 1 \leq 1 + c(1+m)\lambda s x^{1+m}, \tag{19}$$

$$\left(1 - \frac{m x^m}{1 - x^m}\right)^2 < 1. \tag{20}$$

Therefore, we need

$$-1 < 1 - \frac{mx^m}{1-x^m} < 1. \tag{21}$$

The inequality on the right holds, since $\frac{mx^m}{1-x^m} > 0$, for any $m \neq 0$, so we focus on

$$-1 < 1 - \frac{mx^m}{1-x^m}. \tag{22}$$

First, if $m < 0$ then

$$x^m - 1 > 1 - x^m - mx^m, \tag{23}$$
$$(2+m)x^m > 2. \tag{24}$$

Note that $(2+m)x^m$ is decreasing in x for $0 > m > -2$. Since in this region x is bounded above by x^*, $(2+m)x^m$ is bounded below by $(2+m)x^{*m}$. Substituting $x^* = \left(1 - \frac{m\sigma}{c}\right)^{\frac{1}{m}}$ into the above yields

$$(2+m)\left(1 - \frac{m\sigma}{c}\right) > 2, \tag{25}$$
$$m\left(1 - 2\frac{\sigma}{c} - \frac{m\sigma}{c}\right) > 0, \tag{26}$$
$$1 - 2\frac{\sigma}{c} - \frac{m\sigma}{c} < 0, \tag{27}$$
$$\frac{c}{\sigma} < m + 2. \tag{28}$$

Similarly, if $m > 0$ then

$$x^m - 1 < 1 - x^m - mx^m \tag{29}$$
$$(2+m)x^m < 2. \tag{30}$$

Note that $(2+m)x^m$ is increasing in x for $m > 0$. Since in this region x is bounded above by x^*, $(2+m)x^m$ is bounded above by $(2+m)x^{*m}$. Substituting $x^* = \left(1 - \frac{m\sigma}{c}\right)^{\frac{1}{m}}$ into the above yields

$$(2+m)\left(1 - \frac{m\sigma}{c}\right) < 2, \tag{31}$$
$$m\left(1 - 2\frac{\sigma}{c} - \frac{m\sigma}{c}\right) < 0, \tag{32}$$
$$1 - 2\frac{\sigma}{c} - \frac{m\sigma}{c} < 0, \tag{33}$$
$$\frac{c}{\sigma} < m + 2. \tag{34}$$

Thus, $\hat{H}(x, \lambda)$ is concave in x if $m \geq -1$ and $\frac{c}{\sigma} < m + 2$. □

5 Proofs for Hicks-neutral technical change

Note that fixing $\bar{\theta} \equiv \frac{\bar{\sigma}}{\bar{c}}$, defining $\sigma = A\bar{\sigma}, c = A\bar{c}$, and taking derivatives with respect to A is mathematically equivalent to varying c and keeping $\bar{\theta}$ constant by varying σ. For mathematical convenience, we will use this alternative approach to the proofs.

5.1 Proof of $\frac{\mathrm{d}\bar{x}}{\mathrm{d}A} < 0$

Proof. Differentiating $\bar{x} = \bar{q}^{\frac{1}{m}}$ yields

$$\frac{d\bar{x}}{dc} = \frac{1}{m}\bar{q}^{\frac{1}{m}-1}\frac{d\bar{q}}{dc}. \tag{35}$$

Differentiating $\bar{q}$ defined in Eq. (BPT.A.22):

For $m > 0$ the smaller root, $\bar{q}_1$, is the unique stationary root.

$$\begin{aligned}
\frac{d\bar{q}}{dc} &= \frac{1}{2}\frac{m^2 r}{Bc^2} - \frac{1}{4}\left(\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)^2 - 4\left(1 - m\bar{\theta}\right)\right)^{-.5}\left(2\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)\left(\frac{m^2 r}{Bc^2}\right)\right), \\
&= \frac{1}{2}\frac{m^2 r}{Bc^2}\left(1 - \frac{1}{2}\frac{\left(2\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)\right)}{\left(\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)^2 - 4\left(1 - m\bar{\theta}\right)\right)^{.5}}\right), \\
&< \frac{1}{2}\frac{m^2 r}{Bc^2}\left(1 - \frac{1}{2}\frac{\left(2\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)\right)}{\left(\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)^2\right)^{.5}}\right), \\
&= \frac{1}{2}\frac{m^2 r}{Bc^2}\left(1 - \frac{1}{2}\frac{\left(2\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)\right)}{\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)}\right), \\
&= \frac{1}{2}\frac{m^2 r}{Bc^2}\left(1 - 1\right) = 0.
\end{aligned}$$

Hence for $m > 0$, $\frac{d\bar{q}}{dc} < 0$ which, together with Eq. (35), proves $\frac{d\bar{x}}{dc} < 0$.

For $m < 0$ the larger root, $\bar{q}_2$, is the unique stationary root.

$$\begin{aligned}
\frac{d\bar{q}}{dc} &= \frac{1}{2}\frac{m^2 r}{Bc^2} + \frac{1}{4}\left(\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)^2 - 4\left(1 - m\bar{\theta}\right)\right)^{-.5}\left(2\left(2 + m\bar{\theta}\left(m-1\right) - \frac{m^2 r}{Bc}\right)\left(\frac{m^2 r}{Bc^2}\right)\right), \\
&= \frac{1}{2}\frac{m^2 r}{Bc^2} + \frac{1}{4}\left(\left(2\left(1 + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right) + \left(1 - \frac{m\sigma}{c}\right)\right)\right)^2 - 4\left(1 - \frac{m\sigma}{c}\right)\right)^{-.5} \\
&\quad \times\left(2\left(1 + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right) + \left(1 - \frac{m\sigma}{c}\right)\right)\left(\frac{m^2 r}{Bc^2}\right)\right)
\end{aligned}$$

Using results from Appendix BPT.B.1, $\left(1 + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right) + \left(1 - \frac{m\sigma}{c}\right)\right)$ is the sum of roots, which is positive and $\left(2\left(1 + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right) + \left(1 - \frac{m\sigma}{c}\right)\right)\right)^2 - 4\left(1 - \frac{m\sigma}{c}\right)$ is the discriminant, which is positive. Hence for $m < 0$, $\frac{d\bar{q}}{dc} > 0$ which, together with Eq. (35), proves $\frac{d\bar{x}}{dc} < 0$.

Thus $\frac{d\bar{x}}{dc} < 0$ for constant θ, which implies $\frac{\mathrm{d}\bar{x}}{\mathrm{d}A} < 0$. □

5.2 Proof of $\frac{d\bar{s}}{dc} > 0$

The total derivative of $\bar{s}$ with respect to c is

Proof.

$$\frac{d\bar{s}}{dc} = \frac{\partial \bar{s}}{\partial c} + \frac{\partial \bar{s}}{\partial \bar{q}}\frac{d\bar{q}}{dc} \tag{36}$$

Calculating $\frac{\partial \bar{s}}{\partial \bar{q}}$ shows that

$$\bar{s} = \frac{B - \frac{rm}{c}(1-\bar{q})^{-1}}{\left(1 + m\bar{q}(1-\bar{q})^{-1}\right)} = \frac{B\left(\frac{1}{m}(1-\bar{q})\right) - \frac{r}{c}}{\frac{1}{m}(1-\bar{q}) + \bar{q}} > 0, \tag{37}$$

$$\frac{\partial \bar{s}}{\partial \bar{q}} = -\frac{1}{c}\frac{m}{(m\bar{q} - \bar{q} + 1)^2}(r + Bc - mr). \tag{38}$$

Thus,

$$\frac{d\bar{s}}{dc} = \frac{\partial \bar{s}}{\partial c} + \frac{\partial \bar{s}}{\partial \bar{q}}\frac{d\bar{q}}{dc}, \tag{39}$$

$$= \frac{\frac{r}{c^2}}{\frac{1}{m}(1-\bar{q}) + \bar{q}} - \frac{1}{c}\frac{m}{(m\bar{q} - \bar{q} + 1)^2}(r + Bc - mr)\frac{d\bar{q}}{dc}, \tag{40}$$

$$= \frac{1}{c^2}\frac{rm}{(1-\bar{q}) + m\bar{q}} - \frac{1}{c}\frac{m}{(m\bar{q} - \bar{q} + 1)^2}(r + Bc - mr)\frac{d\bar{q}}{dc}, \tag{41}$$

$$= \frac{1}{c}\frac{m}{(m\bar{q} - \bar{q} + 1)}\left(\frac{r}{c} - \frac{(r + Bc - mr)}{(1-\bar{q}) + m\bar{q}}\frac{d\bar{q}}{dc}\right). \tag{42}$$

There is no $\bar{\theta}$ in the equation for $\bar{s}$ (Eq. BPT.37), but this productivity ratio is built into the $\frac{d\bar{q}}{dc}$ term by taking $\theta = \frac{\sigma}{c}$ constant.

From Appendix 5.1, if $m > 0$, then $\bar{q} < 1$ and $\frac{d\bar{q}}{dc} < 0$. Then,

$$\frac{d\bar{s}}{dc} > 0 \text{ if } 0 < m \leq 1, \text{ or better, if } r(1-m) + Bc > 0. \tag{43}$$

From Appendix 5.1, if $m < 0$, then $\bar{q} > 1$ and $\frac{d\bar{q}}{dc} > 0$. Then $(1-\bar{q}) + m\bar{q} < 0$. So,

$$\frac{d\bar{s}}{dc} > 0. \tag{44}$$

□

6 Comparative dynamics for m

6.1 $\bar{q}(m)$ is decreasing

To find $\frac{d\bar{q}}{dm}$, take the total derivative of Eq. (BPT.A.21) with respect to $\bar{q}$ and m:

$$2\bar{q}d\bar{q} - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)d\bar{q} - \left(-\frac{\sigma}{c} + \frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\right)\bar{q}dm - \frac{\sigma}{c}dm = 0,$$

$$\left(2\bar{q} - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)\right)d\bar{q} - \left(\left(-\frac{\sigma}{c} + \frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\right)q + \frac{\sigma}{c}\right)dm = 0.$$

Reorganizing shows that

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}(1-\bar{q})}{2\bar{q} - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)}. \tag{45}$$

Lemma 2. $\frac{d\bar{q}}{dm} \leq 0$

Proof. We analyze the cases of $m > 0$ and $m < 0$ separately.

Case $m > 0$: The unique stable root is $\bar{q}_1$.

$$\bar{q}_1 = \frac{1}{2}\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right) - \left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}\right),$$

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}\left(1 - \bar{q}\right)}{\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right) - \left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}\right) - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)},$$

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}\left(1 - \bar{q}\right)}{-\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}} < 0.$$

since, as we have shown, the discriminant $\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right) > 0$.

Case $m < 0$: The unique stable root is $\bar{q}_2$.

$$\bar{q}_2 = \frac{1}{2}\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right) + \left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}\right),$$

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}\left(1 - \bar{q}\right)}{2\bar{q} - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)},$$

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}\left(1 - \bar{q}\right)}{\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right) + \left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}\right) - \left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)},$$

$$\frac{d\bar{q}}{dm} = \frac{\frac{2m}{c}\left(\sigma - \frac{r}{B}\right)\bar{q} + \frac{\sigma}{c}\left(1 - \bar{q}\right)}{\left(\left(2 - \left(\frac{m\sigma}{c}\right) + \frac{m^2}{c}\left(\sigma - \frac{r}{B}\right)\right)^2 - 4\left(1 - \left(\frac{m\sigma}{c}\right)\right)\right)^{0.5}} < 0,$$

since, in this case, $m < 0$ and $\bar{q} > 1$. □

6.2 Parameter restrictions for $\frac{\mathrm{d}\bar{x}}{\mathrm{d}m}$

Now consider $\bar{x} = \bar{q}^{\frac{1}{m}}$ noting that $0 < \bar{x} < 1$. The derivative of $\bar{X}$ with respect to m is

$$\begin{aligned} \frac{d\bar{x}}{dm} &= -\frac{1}{m^2}\bar{q}^{\frac{1}{m}}\ln\bar{q} + \frac{1}{m}\bar{q}^{\frac{1}{m}-1}\frac{d\bar{q}}{dm}, \\ \frac{d\bar{x}}{dm} &= \frac{1}{m^2}\bar{q}^{\frac{1}{m}}\left(\frac{m}{\bar{q}}\frac{d\bar{q}}{dm} - \ln\bar{q}\right). \end{aligned} \tag{46}$$

As can be seen from Eq. (46), if the following inequality holds, then $\bar{x}(m)$ is decreasing in m:

$$0 > \frac{m}{\bar{q}}\frac{d\bar{q}}{dm} - \ln\bar{q}. \tag{47}$$

Here, we can use the analytic expressions for $\frac{d\bar{q}}{dm}$ from Eq. (45) and $\bar{q}$ from Eq. (BPT.A.22).

We suspect, but have not been able to analytically show, that Assumptions 1, 2, and 3 imply this restriction always holds. Numerically, this inequality constraint has been satisfied for all of the parameter values we have tried that satisfy Assumptions 1, 2, and 3.

6.3 Parameter restrictions for $\frac{\mathrm{d}\bar{s}}{\mathrm{d}m}$

We want to calculate the derivative of $\bar{s}$ with respect to m. Since $\bar{s}$ is a function of other equilibrium values,

$$\frac{d\bar{s}}{dm} = \frac{\partial \bar{s}}{\partial m} + \frac{\partial \bar{s}}{\partial \bar{q}} \frac{d\bar{q}}{dm}. \tag{48}$$

These partials are taken using the expression for $\bar{s}(m)$ provided in Eq. (BPT.37):

$$\frac{\partial \bar{s}}{\partial m} = \frac{1}{c}\left(r + Bc\bar{q}\right) \frac{\bar{q} - 1}{\left(m\bar{q} - \bar{q} + 1\right)^2}, \tag{49}$$

$$\frac{\partial \bar{s}}{\partial \bar{q}} = -\frac{1}{c} \frac{m}{\left(m\bar{q} - \bar{q} + 1\right)^2} \left(r + Bc - mr\right). \tag{50}$$

An expression for $\frac{\mathrm{d}\bar{q}}{\mathrm{d}m} < 0$ is given in Lemma 2. Combining the elements of the total derivative in Eq. (48) yields,

$$\frac{d\bar{s}}{dm} = \frac{1}{c}\left(r + Bc\bar{q}\right) \frac{\bar{q} - 1}{\left(m\bar{q} - \bar{q} + 1\right)^2} - \left(\frac{1}{c} \frac{m}{\left(m\bar{q} - \bar{q} + 1\right)^2} \left(r\left(1 - m\right) + Bc\right)\right) \frac{d\bar{q}}{dm}. \tag{51}$$

Hence, we can get a parameter constraint on $\frac{d\bar{s}}{dm} < 0$ by substituting for $\bar{q}$ from Eq. (BPT.A.22) and for $\frac{\mathrm{d}\bar{q}}{\mathrm{d}m}$ from Eq. (45) into the following inequality:

$$0 < \frac{1}{c}\left(r + Bc\bar{q}\right) \frac{\bar{q} - 1}{\left(m\bar{q} - \bar{q} + 1\right)^2} - \left(\frac{1}{c} \frac{m}{\left(m\bar{q} - \bar{q} + 1\right)^2} \left(r\left(1 - m\right) + Bc\right)\right) \frac{d\bar{q}}{dm}. \tag{52}$$

We suspect, but have not been able to analytically show, that Assumptions 1, 2, and 3 imply this restriction always holds. Numerically, this inequality constraint has been satisfied for all of the parameter values we have tried that satisfy Assumptions 1, 2, and 3.