Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier Online Computational Appendix

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September 27, 2019

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A Numerical Methods

The numerical methods are described in detail, as they are generally applicable to heterogeneous agent models. The structure is as follows: (1) Appendix A.2 collects all of the model equations; (2) Appendix A.3 summarizes the general approach; and (3) Appendix A.4 explicitly maps all of the collected equations to be used by the algorithm;

A.1 Additional Derivations and Bounds

Define the integrated marginal utility from 0 to z as,

$$\hat{w}_i(z) \equiv \int_0^z v_i'(\hat{z})\mathrm{d}\hat{z} = \int_0^z e^{\hat{z}} w_i(\hat{z})\mathrm{d}\hat{z} \tag{A.1}$$

Integrate Technical (B.1) with the initial value from Main (31) and use (A.1) to get,

$$v_i(z) = v(0) + \hat{w}_i(z) \tag{A.2}$$

[1] References to equations, etc. in the main paper are prefixed by *Main* and those to the technical appendix are prefixed *Technical*. An earlier version of this paper was circulated under the title "The Growth Dynamics of Innovation, Diffusion, and the Technology Frontier."

Substitute (A.2) and Technical (B.24) into Technical (A.55) and rearrange to get an expression for $v(0)$ in terms of $\hat{w}_\ell$ and intrinsics,

$$v(0) = \frac{1 + \eta v_\ell(\bar{z})}{r - g + \eta} = \frac{1 + \eta \hat{w}_\ell(\bar{z})}{r - g} \tag{A.3}$$

Value Matching with κ: Keeping in mind the generalization of drawing from $F(z)^\kappa$ for $\kappa > 0$, A change to $w_i(z)$ space will also be useful for simplifying integrals. Note that,[2]

$$\int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z)^\kappa = v(0) + \int_0^{\bar{z}} e^z w_\ell(z) \left(1 - F(z)^\kappa\right) \mathrm{d}z \tag{A.5}$$

And expanding when $\bar{z} < \infty$,

$$\int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z)^\kappa = v_\ell(\bar{z}) - \int_0^{\bar{z}} e^z w_\ell(z) F(z)^\kappa \mathrm{d}z \tag{A.6}$$

Take the value-matching condition in Main (31)

$$v(0) = \int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z)^\kappa - \frac{\zeta}{\psi} \tag{A.7}$$

Use (A.2) and (A.5)

$$v(0) = v(0) + \int_0^{\bar{z}} e^z w_\ell(z)(1 - F(z)^\kappa) \mathrm{d}z - \frac{\zeta}{\psi} \tag{A.8}$$

Simplify,

$$0 = \int_0^{\bar{z}} e^z w_\ell(z)(1 - F(z)^\kappa) \mathrm{d}z - \frac{\zeta}{\psi} \tag{A.9}$$

$$\tag{A.10}$$

[2] These come out of using integration by parts on the calculation of the expectation. For example, if $F(z)$ is the CDF for a random variable Z with minimum and maximum support $\underline{z}$ and $\bar{z}$, then the following holds for any reasonable $h(z)$,

$$\mathbb{E}\left[h(Z)\right] = \int_{\underline{z}}^{\bar{z}} h'(z)(1 - F(z)) \mathrm{d}z + h(\underline{z}) \tag{A.4}$$

KFE From Main (36), for $z < \bar{z}$ the KFE is,[3]

$$0 = gF_\ell'(z) + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - \eta F_\ell(z) + (1-\theta)(S_\ell + S_h)F(z)^\kappa - S_\ell \tag{A.14}$$

In the limit as $z \to \bar{z}$, we know both $\gamma(z)$ and $F(z)$ are continuous. Assume that $\lim_{z\to\bar{z}} F_h'(z) < \infty$ and use $g - \gamma(\bar{z}) = 0$ in Main (15) to (17) and (37) to get the system of equations

$$0 = (\lambda_h + \eta)F_h(\bar{z}) - \lambda_\ell F_\ell(\bar{z}) + gF_h'(0) \tag{A.15}$$
$$1 = F_\ell(\bar{z}) + F_h(\bar{z}) \tag{A.16}$$

Solve to find a relationship between the boundary condition for $F(\bar{z})$ and a given $F'(0)$

$$F_\ell(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\left(gF_h'(0) + \eta + \lambda_h\right) \tag{A.17}$$
$$F_h(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\left(-gF_h'(0) + \lambda_\ell\right) \tag{A.18}$$

From (A.18) , for $F_h(\bar{z}) < 1$, it must be that $F_h'(0) < \lambda_\ell/g$, which provides a bound for possible guesses.

A.2 Nested Summary of Equations for Endogenous Innovation

Summarizing the full set of equations to solve for $F_i(z)$ and $w_i(z)$.
The Hamilton-Jacobi-Bellman Equation (HJBE) for $\kappa = 1$, or for $\psi = 1$ and $\kappa \neq 1$ is

$$0 = w_\ell(0) = w_h(0) \tag{A.19}$$
$$0 = 1 - (r + \lambda_\ell + \eta - (1-\psi)gF'(0))w_\ell(z) - gw_\ell'(z) + \lambda_\ell w_h(z) \tag{A.20}$$
$$0 = 1 - (r + \lambda_h + \eta)w_h(z) - \left(g - \tfrac{\chi}{2}w_h(z)\right)w_h'(z) + \left(\lambda_h + (1-\psi)gF'(0)\right)w_\ell(z) + \tfrac{\chi}{4}w_h(z)^2 \tag{A.21}$$

[3]To account for the drift component, assume a stochastic process with a drift $\mu(z)$, PDF $f(t,z)$, and CDF $F(t,z)$. The KFE comes from the adjoint to infinitesimal generator:[4]

$$\boldsymbol{\partial}_t f(t,z) = -\boldsymbol{\partial}_z\left[\mu(z)f(t,z)\right] + \dots \tag{A.12}$$

Integrating to get the CDF $F(t,z)$, either using the fundamental theorem of calculus or interchanging the order of differentiation and integration, yields

$$\boldsymbol{\partial}_t F(t,z) = -\mu(z)f(t,z) + \dots = -\mu(z)\boldsymbol{\partial}_z F(t,z) + \dots \tag{A.13}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$,

$$0 = 1 - (r + \lambda_\ell + \eta - (1-\psi)\kappa g F'(0) F(z)^{\kappa-1}) w_\ell(z) - g w_\ell'(z) + \lambda_\ell w_h(z) + (1-\psi)\kappa(\kappa-1) g F'(0) F(z)^{\kappa-2} F'(z) e^{-z} \int_0^z w_\ell(\hat{z}) \mathrm{d}\hat{z} \tag{A.22}$$

$$0 = 1 - (r + \lambda_h + \eta) w_h(z) - \left(g - \tfrac{\chi}{2} w_h(z)\right) w_h'(z) + \left(\lambda_h + (1-\psi)\kappa g F'(0) F(z)^{\kappa-1}\right) w_\ell(z) + \tfrac{\chi}{4} w_h(z + (1-\psi)\kappa(\kappa-1) g F'(0) F(z)^{\kappa-2} F'(z) e^{-z} \int_0^z w_\ell(\hat{z}) \mathrm{d}\hat{z} \tag{A.23}$$

The innovation rate is,

$$\gamma(z) = \tfrac{\chi}{2} w_h(z) \tag{A.24}$$

In either case, $\bar{z}$ and g are related through,

$$g \equiv \gamma(\bar{z}) \tag{A.25}$$

The KFE can be solved as an initial value problem (or boundary value problem depending on the algorithm) subject to value matching,

$$0 = F_\ell(0) = F_h(0) \tag{A.26}$$

$$0 = g F_\ell'(z) + \lambda_h F_h(z) - (\lambda_\ell + \eta) F_\ell(z) + (1-\theta) g F'(0) F(z)^\kappa - g F_\ell'(0) \tag{A.27}$$

$$0 = (g - \gamma(z)) F_h'(z) + \lambda_\ell F_\ell(z) - (\lambda_h + \eta) F_h(z) - g F_h'(0) \tag{A.28}$$

$$0 = -\frac{1}{\psi}\left(\zeta + \tfrac{1}{\vartheta}\theta^2 + \tfrac{1}{\varsigma}\kappa^2\right) + \int_0^{\bar{z}} e^z w_\ell(z) \mathrm{d}z - (1-\theta) \int_0^{\bar{z}} e^z w_\ell(z) F(z)^\kappa \mathrm{d}z \tag{A.29}$$

Note that if $\kappa = 1$, the KFE is a linear system of ODEs. More generally, the endogenous κ must solve the implicit equation,

$$\kappa = \frac{-\varsigma\psi(1-\theta)}{2} \int_0^{\bar{z}} e^z w_\ell(z) \log(F(z)) F(z)^\kappa \mathrm{d}z \tag{A.30}$$

Similarly, the following determines the endogenous θ equation

$$\theta = \frac{\psi\vartheta}{2} \int_0^{\bar{z}} e^z w_\ell(z) F(z)^\kappa \mathrm{d}z \tag{A.31}$$

KFE in PDFs with $\kappa = 1$ Take (A.27) and (A.28) and differentiate to find the KFE in the PDFs when $\kappa = 1$ and $f_i(z) \equiv \boldsymbol{\partial}_z F_i(z)$,

$$0 = g f_\ell'(z) + \lambda_h f_h(z) - (\lambda_\ell + \eta) f_\ell(z) + (1-\theta) g f(0) f(z) \tag{A.32}$$

$$0 = (g - \gamma(z)) f_h'(z) + \lambda_\ell f_\ell(z) - (\lambda_h + \eta + \gamma'(z)) f_h(z) \tag{A.33}$$

Verification and Guesses In the general case, given a $F'(0)$, one could solve the quadratic in (A.34) and (A.35) for c_ℓ and c_h and then use (A.36) to find the maximum g as $\bar{z} \to \infty$,[5]

$$0 = 1 - (r + \lambda_\ell + \eta - (1-\psi)\tfrac{\chi}{2} c_h F'(0))c_\ell + \lambda_\ell c_h \tag{A.34}$$

$$0 = 1 - (r + \lambda_h + \eta)c_h + \left(\lambda_h + (1-\psi)\tfrac{\chi}{2} c_h F'(0)\right) c_\ell + \tfrac{\chi}{4} c_h^2 \tag{A.35}$$

$$g_{\max} = \tfrac{\chi}{2} c_h \tag{A.36}$$

In the case of $\psi = 1$, then the maximum g is independent of $F'(0)$ and comes from

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell} \tag{A.37}$$

$$g < \bar{\lambda}(r+\eta)\left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r+\eta)^2}}\right] \tag{A.38}$$

While not necessary to solve for the equilibrium, the non-stationary value functions can be calculated with

$$v(0) \equiv \frac{1 + \eta \int_0^{\bar{z}} e^z w_\ell(z) \mathrm{d}z}{r - g} \tag{A.39}$$

$$v_i(z) = v(0) + \int_0^z e^{\hat{z}} w_i(\hat{z}) \mathrm{d}\hat{z} \tag{A.40}$$

The mass of high and low firms fulfills,

$$1 = F_\ell(\bar{z}) + F_h(\bar{z}) \tag{A.41}$$

$$F_\ell(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\left(g F_h'(0) + \eta + \lambda_h\right) \tag{A.42}$$

$$F_h(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\left(-g F_h'(0) + \lambda_\ell\right) \tag{A.43}$$

Note, from (A.41) and (A.43) that the maximum guess on $F_h'(0)$ is,

$$F_h'(0) < \frac{\lambda_\ell}{g} \tag{A.44}$$

An alternative approach for verifying guesses is to start with $F_h(\bar{z}) \in (0,1)$ and use

$$F_h'(0) = \frac{\lambda_\ell - (\eta + \lambda_h + \lambda_\ell)F_h(\bar{z})}{g} \tag{A.45}$$

[5] Since we have put these equations into the general collocation system, it is not necessary to solve this for a particular $\bar{z}$. To find $g_{\max}$ numerically, just choose a large $\bar{z}$.

From the properties of the solution, we also know the following constraints,

$$w_i(z) \geq 0 \tag{A.46}$$

$$w_i'(z) \geq 0 \tag{A.47}$$

And to be a valid CDF,

$$F_i(z) \geq 0 \tag{A.48}$$

$$F_i'(z) \geq 0 \tag{A.49}$$

A.3 Summary of Spectral Solution Method

The general structure of the problem is a set of ODEs with parameters constrained by equilibrium conditions (themselves functions of the solutions to the ODEs). A natural approach is to approximation the $w_i(z)$ and $F_i(z)$ functions some finite dimensional polynomial, and find the coefficients that fulfill all of the ODEs and equilibrium conditions—i.e., every equation in Appendix A.2. This general approach is called a spectral method, and we will concentrate on the spectral collocation method (i.e., solve a nonlinear system in the coefficients on the function basis with a matching number of equations). Alternatively, this could be done with the related Weighted Residual methods, over-constraining the system of equations and finding the minimum residual using nonlinear least squares, etc.[6] To summarize the approach used:

1. Pick a $\bar{z}$. In the case of unbounded support, this should be a large number to ensure convergence. In the case of bounded support, these will parameterize the set of stationary equilibrium (though there would not be a feasible for every $\bar{z}$).

2. Choose a Chebyshev basis for the $w_i(z)$ and $F_i(z)$ adapted to $z \in [0, \bar{z}]$.[7]

[6] In the special case of $\psi = 1$ with g known, the problems are decoupled. In that case, the HJBE can be solved with a stiff ODE solver such as Matlab's `ode15s` as long as the solution terminates as it approaches the singularity. To accomplish this, stop at the $\bar{z}$ such that $g - \frac{\chi}{2}w_h(\bar{z}) \approx 0$. For stability choose a small threshold value, but ensure that at the chosen $\bar{z}$, $\frac{\chi}{2}w_h(\bar{z}) < g$, which will ensure that singularities don't occur in subsequent calculations and cause havoc or divergence. In matlab, see ODE settings of 'Events'. See `http://www.mathworks.com/help/matlab/ref/odeset.html#f92-1017470`.

[7] An alternative consideration for the unbounded case is a Gauss-Laguerre basis. For relatively large $\bar{z}$, the benefit of Gauss-Laguerre is that the quadrature nodes will calculate $\int_0^\infty h(z)\mathrm{d}z$ rather than using $\int_0^{\bar{z}} h(z)\mathrm{d}z$ to approximate the integral. Furthermore, the nodes of the polynomial are spaced closer to the minimum of support. The worry is that since $\bar{z}$ is finite, for the ODE calculations issues such as Runge's Phenomenon in the right tail are not minimized—while Chebyshev polynomials provide the best convergence for these artifacts at corners. Hence, to minimize errors and nest the unbounded and bounded cases, we chose Chebyshev polynomials. While we see some oscillations at the corners, with Chebyshev basis the errors seem to be within tolerance bounds.

3. Calculate the collocation nodes (i.e., the roots of the basis polynomials) and the quadrature weights for the particular basis. For example, if using Chebyshev basis, then the nodes would be the Gauss-Chebyshev quadrature weights.

   Aligning the quadrature and collocation nodes is the key step.

4. Setup a system of all the equations, ODEs, and boundary conditions

   - For the fixed $\bar{z}$, the set of equations are for $\{w_\ell(z), w_h(z), F_\ell(z), F_h(z), \kappa, \theta\}$ (where the functions are parameterized by a finite dimensional set of coefficients)
   - The complete set of equations in Appendix A.2 must hold, where the functional equations must hold at all of the collocation node points.
   - A key trick here is that we are evaluating the $w_i(z)$ and $F_i(z)$ at the same nodes as we would use for the associated quadrature. This means that with the appropriate Gaussian quadrature weights, we can calculate integrals such as $\int_0^{\bar{z}} w_\ell(z)(1 - F(z))\mathrm{d}z$ as a simple linear function. While this general approach significantly simplifies the calculations, its main purpose is to enable auto-differentiation for solvers to use the Jacobian of the residual.

5. Use a nonlinear solver to find the coefficients of the $w_i(z)$ and $F_i(z)$, along with any other variables such as g, κ, or θ.

   - To solve systems of this size, it is necessary to find the Jacobian of the system of equations. This can be accomplished easily using auto-differentiation. Furthermore, the Jacobian is also somewhat sparse, so specialized algorithms that exploit sparsity are helpful.
   - Since the number of variables is large (e.g., if 100 nodes are used per function basis, this is 402 variables), a good solver is necessary. We are using the NLSSOL constrained nonlinear least squares solver from `http://tomopt.com/tomlab/products/npsol/solvers/NLSSOL.php`, which has built in auto-differentiation. With this, we can find the complete solution in $\approx$ 2-20 seconds with 100 nodes per function.

A.4 Joint Spectral Collocation Algorithm

Notation For a vectors x, y and scalar a: Denote the scalar-vector products as ax, the dot-product (or matrix-vector product) as $x \cdot y$, the point-wise product of vectors as $x \odot y$. For the j'th element denote it $x(j)$ and a slice between j and k as $x(j : k)$ (where the first index is 0).

For vectors that are only on the interior nodes (i.e., for quadrature), these are denoted such as $e^{\vec{z}}_{\text{int}} \equiv e^{\vec{z}}(1 : N)$ or $\vec{z}_{\text{int}} \equiv \vec{z}(1 : N)$

Parameters The complete set of model parameters for the nested model is $\{r, \lambda_\ell, \lambda_h, \eta, \psi, \chi, \zeta, \vartheta, \varsigma\}$, and if they are not endogenous, then an additional $\left\{\bar{\theta}, \bar{\kappa}\right\}$ is used (and the corresponding $\{\vartheta, \varsigma\}$ are not necessary). If the consumer's IES is used for discounting, then r is determined in equilibrium from ρ and Λ.

In addition, $\bar{z}$ is effectively a fixed parameter for the calculations. It is set large (e.g., $\bar{z} = 9.3$, which corresponds to $\bar{Z}(t)/M(t) \approx 10000$) in the case of unbounded support, and is set to a fixed number when looking for the set of equilibria for bounded support.

Setup and Function Approximation Given a $\bar{z}$, define the approximations with $N-1$ order Chebyshev polynomials, $T_k(z)$ adapted to the support $[0, \bar{z}]$. Then, approximation $w_i(z)$ and $F_i(z)$ through,

$$w_i(z) \approx \sum_{n=0}^{N-1} c_{\text{in}} T_n(z) \tag{A.50}$$

$$F_i(z) \approx \sum_{n=0}^{N-1} d_{\text{in}} T_n(z) \tag{A.51}$$

Denote the vectors of coefficients as $d_i \in \mathbb{R}^N$ and $c_i \in \mathbb{R}^N$. Define the Chebyshev polynomial roots,

$$\vec{z}_{\text{int}} \equiv \{z_1, \dots z_N\} \in \mathbb{R}^N \tag{A.52}$$

And the complete set of nodes including boundary values as (with $z_0 \equiv 0$ and $z_{N+1} \equiv \bar{z}$)

$$\vec{z} \equiv \{0, z_1, \dots z_N, z_N, \bar{z}\} \in \mathbb{R}^{N+2} \tag{A.53}$$

Define the basis matrices for these nodes, B, by stacking stacking the polynomials evaluated at $\vec{z}$, Similarly, define B' as the basis of the first derivative, and $\breve{B}$ as the basis of the partial integrals.[8]

$$B \equiv \begin{bmatrix} T_0(z_0) & \dots & T_{N-1}(z_0) \\ \dots & & \dots \\ T_0(z_{N+1}) & \dots & T_{N-1}(z_{N+1}) \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{A.55}$$

$$B' \equiv \begin{bmatrix} T_0'(z_0) & \dots & T_{N-1}'(z_0) \\ \dots & & \dots \\ T_0'(z_{N+1}) & \dots & T_{N-1}'(z_{N+1}) \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{A.56}$$

$$\breve{B} \equiv \begin{bmatrix} \int_0^{z_0} T_0(z)\mathrm{d}z & \dots & \int_0^{z_0} T_{N-1}(z)\mathrm{d}z \\ \dots & & \dots \\ \int_0^{z_{N+1}} T_0(z)\mathrm{d}z & \dots & \int_0^{z_{N+1}} T_{N-1}(z)\mathrm{d}z \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{A.57}$$

With these, the functions and derivatives evaluated at all $\vec{z}$ are a simple matrix-vector product,

$$\vec{w}_i \equiv \{w_i(z_n)\}_{n=0}^{N+1} = B \cdot c_i \in \mathbb{R}^{N+2} \tag{A.58}$$

$$\vec{w}_i' \equiv \{w_i'(z_n)\}_{n=0}^{N+1} = B' \cdot c_i \in \mathbb{R}^{N+2} \tag{A.59}$$

$$\breve{w}_i \equiv \left\{\int_0^{z_n} w_i(z)\mathrm{d}z\right\}_{n=0}^{N+1} = \breve{B} \cdot c_i \in \mathbb{R}^{N+2} \tag{A.60}$$

$$\vec{F}_i \equiv \{F_i(z_n)\}_{n=0}^{N+1} = B \cdot d_i \in \mathbb{R}^{N+2} \tag{A.61}$$

$$\vec{F}_i' \equiv \{F_i'(z_n)\}_{n=0}^{N+1} = B' \cdot d_i \in \mathbb{R}^{N+2} \tag{A.62}$$

Given Chebyshev quadrature weights $\omega \in \mathbb{R}^N$ with $\vec{h}_{\mathrm{int}} \equiv \{h(z) | z \in \vec{z}_{\mathrm{int}}\} \in \mathbb{R}^N$, integrals

[8] With `compecon`, the basis can be calculated for nodes `all_nodes` and function space `fspace` as `B = funbas(fspace, vec_z_all, 0)`. The basis of the derivative is simply `B_p = funbas(fspace, vec_z_all, 1)`. With compecon, to calculate the quadrature weights for Chebyshev quadrature use `qnwcheb(N,0,z_bar)`. The basis for the integrals doesn't appear to be in compecon, but can be calculated using the recurrence formulas for Chebyshev polynomials (`https://en.wikipedia.org/wiki/Chebyshev_polynomials#Differentiation_and_integration`). For the $\tilde{T}_n(x)$ polynomials on $[-1,1]$, the formula is $\int \tilde{T}_n(x)\mathrm{d}x = \frac{1}{2}\left(\frac{\tilde{T}_{n+1}(x)}{n+1} - \frac{\tilde{T}_{n-1}(x)}{n-1}\right)$. When adapted to $[0, \bar{z}]$, this becomes

$$\int_0^z T_n(\tilde{z})\mathrm{d}\tilde{z} = \frac{\bar{z}}{2}\left(\frac{T_{n+1}(z)}{n+1} - \frac{T_{n-1}(z)}{n-1}\right) - \frac{\bar{z}}{2}\left(\frac{T_{n+1}(0)}{n+1} - \frac{T_{n-1}(0)}{n-1}\right) \tag{A.54}$$

where for $z_j \in \vec{z}$ the $T_n(z_j)$ can be calculated from B.

over the entire domain are a simple vector product for any $h(z)$,[9]

$$\int_0^{\bar{z}} h(z)\mathrm{d}z \approx \omega \cdot \vec{h}_{\text{int}} \tag{A.63}$$

Approximate Problem With the finite dimensional approximation, the complete set of guesses to calculate a root to the system of equations is,

$$x \equiv \{c_0, c_1, \dots c_{N-1}, d_0, d_1, \dots d_{N-1}, \kappa, \theta\} \in \mathbb{R}^{4N+2} \tag{A.64}$$

Define a residual operator $\mathcal{L} : \mathbb{R}^{4N+2} \to \mathbb{R}^{\hat{N}}$. The solution is an x^* such that $\mathcal{L}(x^*) \approx 0$. If this was square, then $\hat{N} = 4N + 2$. Add linear inequality and equality constraints with matrix $\Psi \in \mathbb{R}^{\hat{N}_{\text{con}} \times (4N+2)}$ and vectors $\bar{b}, \underline{b}$ in $\mathbb{R}^{\hat{N}_{\text{con}}}$ and box-bounds $\bar{x}, \underline{x} \in \mathbb{R}^{4N+2}$,

$$\begin{aligned} x^* = \arg\min_x & \left\{\frac{1}{2}\mathcal{L}(x) \cdot \mathcal{L}(x)\right\} \\ \text{s.t.}\ & \underline{b} \le \Psi \cdot x \le \bar{b} \\ & \underline{x} \le x \le \bar{x} \end{aligned} \tag{A.65}$$

Summary of Pre-calculations At the end of the pre-calculations, we have a $\{\vec{z}, B, B', \omega, \Omega\}$ to be used in further calculations. Insofar as there is a "trick" to the algorithm, it is that the quadrature and collocation schemes are chosen so the nodes line up, which means that the functions only need to be calculated at a single set of nodes for both the ODE and integral equilibrium conditions.

Calculating the Residual Given the x and using the fixed B, B', and $\vec{z}$, calculate the following:

Setup in $\mathcal{L}$ Calculate $\vec{F}_\ell, \vec{F}_h, \vec{w}_\ell, \vec{w}_h, \breve{w}_\ell, \vec{F}'_\ell, \vec{F}'_h, \vec{w}'_\ell, \vec{w}'_h$, and (A.24) and (A.25),

$$F'(0) = \vec{F}'_\ell(0) + \vec{F}'_h(0) \in \mathbb{R} \tag{A.66}$$

$$\vec{\gamma} = \tfrac{\chi}{2}\vec{w}_h \in \mathbb{R}^{N+2} \tag{A.67}$$

$$g = \vec{\gamma}(N+1) \in \mathbb{R} \tag{A.68}$$

$$r = \rho + \Lambda g \in \mathbb{R}, \quad \text{if using consumer with CRRA preferences} \tag{A.69}$$

$$\vec{F}^\kappa = (\vec{F}_\ell + \vec{F}_h)^\kappa \in \mathbb{R}^{N+2}, \quad \text{i.e., pointwise power.} \tag{A.70}$$

[9]Note that the end-points are not used in the Gaussian quadrature formula. To calculate the quadrature weights for Chebyshev quadrature in compecon use `qnwcheb(N - 1,0,z_bar)`

The following are only defined for interior nodes,

$$\tilde{w}_{\ell,\text{int}} = e^{\vec{z}}_{\text{int}} \odot \vec{w}_{\ell,\text{int}} \in \mathbb{R}^N \tag{A.71}$$

Equation for ℓ HJBE From (A.20). This provides a vectorized calculation of all residuals.

$$0 \approx 1 - (r + \lambda_\ell + \eta - (1-\psi)gF'(0))\vec{w}_\ell - g\vec{w}'_\ell + \lambda_\ell \vec{w}_h \tag{A.72}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$, from (A.22),

$$\begin{aligned} 0 \approx\ & 1 - (r + \lambda_\ell + \eta - (1-\psi)\kappa g F'(0)\, \vec{F}^{\kappa-1}) \odot \vec{w}_\ell - g\vec{w}'_\ell + \lambda_\ell \vec{w}_h \\ & + (1-\psi)\kappa(\kappa-1)gF'(0)\, \vec{F}^{\kappa-2} \odot (\vec{F}'_\ell + \vec{F}'_h) \odot e^{-\vec{z}} \odot \breve{w}_\ell \end{aligned} \tag{A.73}$$

Equation for h HJBE From (A.21). As discussed in Appendix A.3, it may be preferable to split the $\vec{\gamma}$ and $\vec{w}_h$ to create a DAE, and add in the equations separately.

$$0 \approx 1 - (r + \lambda_h + \eta)\vec{w}_h - (g - \vec{\gamma}) \odot \vec{w}'_h + (\lambda_h + (1-\psi)gF'(0))\, \vec{w}_\ell + \tfrac{\chi}{4}\vec{w}_h \odot \vec{w}_h \tag{A.74}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$, from (A.23),

$$\begin{aligned} 0 =\ & 1 - (r + \lambda_h + \eta)\vec{w}_h - (g - \vec{\gamma}) \odot \vec{w}'_h + \left(\lambda_h + (1-\psi)\kappa g F'(0)\vec{F}^{\kappa-1}\right) \odot \vec{w}_\ell + \tfrac{\chi}{4}\vec{w}_h \odot \vec{w}_h \\ & + (1-\psi)\kappa(\kappa-1)gF'(0)\, \vec{F}^{\kappa-2} \odot (\vec{F}'_\ell + \vec{F}'_h) \odot e^{-\vec{z}} \odot \breve{w}_\ell \end{aligned} \tag{A.75}$$

Equation for ℓ KFE From (A.27)

$$0 \approx g\vec{F}'_\ell + \lambda_h \vec{F}_h - (\lambda_\ell + \eta)\vec{F}_\ell + (1-\theta)gF'(0)\vec{F}^{\kappa} - gF'_\ell(0) \tag{A.76}$$

Equation for h KFE From (A.28)

$$0 \approx (g - \vec{\gamma}) \odot \vec{F}'_h + \lambda_\ell \vec{F}_\ell - (\lambda_h + \eta)\vec{F}_h - gF'_h(0) \tag{A.77}$$

Equations for initial conditions From (A.19) and (A.26),

$$0 \approx F_\ell(0) \tag{A.78}$$

$$0 \approx F_h(0) \tag{A.79}$$

$$0 \approx w_\ell(0) \tag{A.80}$$

$$0 \approx w_h(0) \tag{A.81}$$

Equation for Value Matching Since we are calculating for a finite $\bar{z}$, the general case of (A.29) nests the unbounded support for a 'large' $\bar{z}$. In the case of exogenously given θ or κ, then let ς and/or $\vartheta = \infty$ to ensure they don't add to the cost.

$$0 \approx -\frac{1}{\psi}\left(\zeta + \tfrac{1}{\vartheta}\theta^2 + \tfrac{1}{\varsigma}\kappa^2\right) + \omega \cdot \tilde{w}_{\ell,\text{int}} - (1-\theta)\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] \tag{A.82}$$

Equation for Endogenous κ From (A.30),

$$0 = -\kappa + \begin{cases} \frac{-\varsigma\psi(1-\theta)}{2}\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \log \vec{F}_{\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] & \text{if endogenous} \\ \bar{\kappa} & \text{if fixed at } \bar{\kappa} \end{cases} \tag{A.83}$$

Equation for Endogenous θ From (A.31),

$$0 \approx -\theta + \begin{cases} \frac{\psi\vartheta}{2}\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] & \text{if endogenous} \\ \bar{\theta} & \text{if fixed at } \bar{\theta} \end{cases} \tag{A.84}$$

Equation for $F(\bar{z}) = 1$ From (A.41)

$$0 \approx F_l(\bar{z}) + F_h(\bar{z}) - 1 \tag{A.85}$$

Linear Constraints The linear equality constraints, such as (A.41) have all been added directly to the $\mathcal{L}$ residual. Linear inequality constraints come from (A.45), (A.46), (A.48) and (A.49).[10] As this polynomial basis is not shape preserving, the constraints on $F_i(z)$ and $F_i'(z) > 0$ are unlikely to hold everywhere. Instead, choose a small threshold $\epsilon > 0$ such that $F_i'(z) > -\epsilon$ for all z and $F_i'(z) > -\epsilon$ for all z. The oscillations around endpoints, while minimized with a Chebyshev basis, could cause problems in our algorithm since g is so dependent on $F'(0)$, so $F_i'(0) \geq 0$ are added directly as a special case. While the $w_i(z) > 0$ and $F_i(z) \geq 0$ constraints could also be added with appropriate ϵ, they are left out for simplicity.

Recall that $w_i(0) \approx \bar{Z}(0,:) \cdot c_i$, $F_i(0) \approx \bar{Z}(0,:) \cdot d_i$, etc. Let $\vec{0} \equiv 0_N, \vec{\epsilon} \equiv \epsilon_{N+2}, \vec{\infty} \equiv \infty_N$ and $\vec{\mathbf{0}} \equiv 0_{(N+2)\times N}$

[10] In an interior solution, few of the inequality constraints would be binding and $\epsilon \approx 0$.

$$
\overbrace{\begin{bmatrix} -\vec{\epsilon} \\ -\vec{\epsilon} \\ 0 \\ 0 \end{bmatrix}}^{\underline{b}} \leq \overbrace{\begin{bmatrix} \vec{\mathbf{0}} & \vec{\mathbf{0}} & B' & \vec{\mathbf{0}} & \vec{0} & \vec{0} \\ \vec{\mathbf{0}} & \vec{\mathbf{0}} & \vec{\mathbf{0}} & B' & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & B'(0,:) & \vec{0} & 0 & 0 \\ \vec{0} & \vec{0} & \vec{0} & B'(0,:) & 0 & 0 \end{bmatrix}}^{\Psi} x \leq \overbrace{\begin{bmatrix} \vec{\infty} \\ \vec{\infty} \\ \infty \\ \infty \end{bmatrix}}^{\bar{b}}, \text{ from } \begin{bmatrix} (A.49) \\ (A.49) \\ (A.49) \\ (A.49) \end{bmatrix} \tag{A.86}
$$

With bounds on the x as,

$$
\overbrace{\begin{bmatrix} -\vec{\infty} \\ -\vec{\infty} \\ -\vec{\infty} \\ -\vec{\infty} \\ 0 \\ 0 \end{bmatrix}}^{\underline{x}} \leq x \leq \overbrace{\begin{bmatrix} \vec{\infty} \\ \vec{\infty} \\ \vec{\infty} \\ \vec{\infty} \\ \infty \\ 1 \end{bmatrix}}^{\bar{x}} \tag{A.87}
$$

Mean and Gini Given the Chebyshev collocation nodes and the solution for d_i, the mean and Gini coefficient can be calculated with,

$$
\text{Mean} = \int_0^{\bar{z}} (1 - F(z)) \mathrm{d}z = \omega \cdot (1 - \vec{F}_{\text{int}}) \tag{A.88}
$$

$$
\text{Gini} = 1 - \frac{\int_0^{\bar{z}} ((1 - F(z))^2 \mathrm{d}z}{\int_0^{\bar{z}} (1 - F(z)) \mathrm{d}z} = 1 - \frac{\omega \cdot (1 - \vec{F}_{\text{int}})^2}{\omega \cdot (1 - \vec{F}_{\text{int}})} \tag{A.89}
$$