Risk Markups

Sebastian Di Tella, Stanford GSB & NBER
Cedomir Malgieri, Stanford
Christopher Tonetti, Stanford GSB & NBER

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Optimal policy in an economy with misallocation depends on the origin of markups. We develop a model of heterogeneous markups generated by uninsurable persistent idiosyncratic risk. Entrepreneurs hire labor trading off expected profits against risk. Markups arise as compensation for risk and create misallocation. We study the constrained-efficient allocation of a planner who can use a uniform labor tax and time-zero lump-sum transfers. The optimal keep rate equals the product of (1) the aggregate markup and (2) workers' consumption share divided by their Pareto weight. The markup component reflects inefficient risk premia that could be improved with a labor subsidy. The consumption-share component reflects inefficient precautionary saving that could be improved with a labor tax. In the long-run, the precautionary-saving component dominates and the optimal policy is a tax with a keep rate equal to workers' consumption divided by labor income.

1 Introduction

There is a large literature that measures markups and finds them to be sizable and heterogeneous across firms (Hall (1988), De Loecker and Warzynski (2012)). As Restuccia and Rogerson (2008) and Hsieh and Klenow (2009) emphasize, markups can cause misallocation—i.e., lower TFP. The role for policy to address misallocation depends on the source of the markups. For example, a common view is that measured markups arise because firms have market power. From this perspective, Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023) show that it is optimal to undo the markup distortions by subsidizing firms to increase their production, with larger subsidies for the larger-markup firms.

Since at least Knight (1921) economists have recognized that profits may reflect risk compensation. Boar, Gorea, and Midrigan (2023) quantify the sources of the dispersion in firm profits and returns and attribute the vast majority of the variation to uninsurable idiosyncratic risk, with some contribution from decreasing returns to scale production technology (a potential model stand-in for markups), and little contribution from collateral constraints.

In this paper, we entertain this possibility that measured markups are risk premia. In contrast to market-power based markup settings, we know very little about optimal policy in this environment. The main contribution of this paper is to study constrained efficiency in such an economy. In our environment, we find it is optimal in the long-run to tax labor in order to counteract inefficient precautionary saving. This is in sharp contrast to the optimal policy of labor subsidies in the market-power markup literature.

Model Overview and Main Results. This paper develops a model to study how uninsurable idiosyncratic entrepreneurial risk affects markups, misallocation, and efficiency. We show how a planner who cannot complete markets, but can use taxes, subsidies, and transfers, can improve efficiency and welfare.

In our model, there is a representative worker and a continuum of entrepreneurs. The workers supply labor, which is the only factor of production, and face no risk. Each entrepreneur is endowed with a distinct linear technology to produce the homogeneous final good. Entrepreneurs differ in the productivity of their technology. There are three essential elements of the economic environment. First, conditional expected productivity follows a stationary persistent stochastic process. Because productivity is persistent, the conditional expected productivity is an entrepreneur's type, and having different types allows for the possibility of misallocation of labor across types. Second, hiring labor is risky. Entrepreneurs must hire labor before the realization of their productivity shock. Third, entrepreneurs are risk averse.

We study the competitive equilibrium, in which there are competitive markets for goods, labor, a risk free bond, and firm equity. This allows us to study the role of risk markups in isolation, absent other sources of markups like market power. The fourth essential element of the model is that idiosyncratic risk is not fully insurable. Motivated by moral hazard arguments, the entrepreneur can sell only up to a fraction $1 - \phi$ of the firm value and must retain a ϕ share of the firm equity. Thus, the entrepreneur's firm's profits directly enter his budget constraint.

Due to competitive markets, all firms charge the same price for the homogeneous good they sell and pay the same wage to hire labor. Risk-averse entrepreneurs, risky production, and incomplete risk sharing lead to risk markups. That is, there is a wedge in the first order condition for hiring labor, such that the price of the final good is greater than the unit labor cost. This risk markup wedge is a risk premium. Entrepreneurs hire less labor than they would if they had full insurance. Hiring labor is like investing in a risky asset and they consider the trade off between the expected return and the risk associated with hiring.

The persistent productivity process generates heterogeneity in the risk markup. The labor an entrepreneur hires is increasing in his expected productivity (holding wealth constant). However, even though production of the homogeneous good is linear, the most productive entrepreneur does not hire all the labor because that would expose him to too much risk. There is a sense in which there are no markups in the competitive equilibrium, since the risk-adjusted marginal cost is equal to the price of the final good for all firms. Nonetheless, the risk markup wedge generates misallocation and is reflected in lower TFP.

In this environment we study the constrained efficient allocation. We study a planner who has access to a time-varying tax on workers' wages (instantly redistributed lump sum) and time-zero lump-sum transfers. The lump-sum transfers allow the planner to achieve any desired redistribution motive, allowing us to focus on whether there are efficiency reasons to use the wage tax. The main result of the paper is that the optimal keep rate (one minus the tax rate) is equal to the product of (1) the aggregate markup and (2) workers' consumption share divided by their Pareto weight.

The markup component reflects inefficient risk premia and the consumption-share component reflects inefficient precautionary saving. Even though the risk markup represents compensation for utility-relevant risk taking, it is still inefficient. In isolation, risk premia induce inefficiently low labor demand due to a coordination problem. Individually, as an entrepreneur hires less labor, they are exposed to less risk and consequently have less volatility in their consumption. However, if all entrepreneurs were to hire more labor than in the competitive equilibrium, they would produce more output and that extra output could be used to smooth the extra exposure to idiosyncratic risk, so that consumption is increased while consumption volatility is held constant. To correct this inefficiency in isolation, the planner wants a labor subsidy equal to the aggregate markup.

However, the inefficient risk markup is not the planner's sole concern. Just like in the Aiyagari (1994) model, uninsurable idiosyncratic risk leads to inefficient precautionary saving. Entrepreneurs accumulate too much wealth relative to workers and, since the constrained efficient planner takes Euler equations as a constraint, the entrepreneurs must receive a larger share of consumption than is warranted by their Pareto weights. In the long-run, the optimal policy is a tax with a keep rate equal to workers' consumption divided by labor income. To understand why a dynamic inefficiency results in a labor tax, we show that a planner who can also tax entrepreneurs' saving can achieve the first best by setting a labor *subsidy* equal to the aggregate markup and a saving tax to exactly offset precautionary saving.

1.1 Relationship to Existing Literature

We contribute to the recent literature on risk and misallocation. In David, Schmid, and Zeke (2022), dispersion in the marginal product of capital is generated by heterogeneous risk premia. There is an aggregate shock and different firms have different exogenous loadings on the aggregate risk factor. As in CAPM, the different risk profiles generate different risk premia, which generate deviations from equalizing MPKs, even though the decentralized equilibrium is efficient. Our paper also explores the relationship between risk and misallocation, but in our model risk premia are endogenous, generated by idiosyncratic risk and incomplete markets. This leads to a substantively different perspective on optimal policy. David and Zeke (2024) further explore the implications of aggregate risk and misallocation for optimal monetary policy in a New Keynesian model and David, Ranciere, and Zeke (2023) explores the implications for international diversification and the labor share.

Eeckhout and Veldkamp (2023) is another paper exploring how risk affects our interpretation of measured markups. In their paper, firms acquiring data can reduce risk markups by reducing risk (improving forecasting) but can increase markups by inducing investment to capture market share and exploit market power. Dou, Ji, Tian, and Wang (2024) also studies how risk and incomplete markets affects misallocation, with a focus on how misallocation introduces medium-run fluctuations in TFP that affect aggregate growth and asset prices. These are all in contrast to papers on misallocation without an essential role for risk, such as those that generate misallocation with market power, such as Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023), or adjustment costs, as in Asker, Collard-Wexler, and Loecker (2014).

Boar, Gorea, and Midrigan (2023) is a closely related paper that also studies an environment with uninsurable idiosyncratic risk. Theirs is a quantitative paper that explains the source of heterogeneity in firm returns. Their model additionally features decreasing returns to scale production technology and collateral constraints. They find that uninsurable idiosyncratic risk is by far the most important feature that explains the empirical variation in returns. Motivated by their quantitative results on the importance of uninsurable idiosyncratic risk, we compliment their paper by providing a theoretical analysis of optimal policy in such a world. Since optimal policy is relatively well understood when misallocation is generated by market power or by collateral constraints, we only model uninsurable idiosyncratic risk. This allows us to provide insights from analytical results in a tractable model.

Our model of entrepreneurship is similar to that in Moll (2014) and our analysis of optimal policy is similar to Itskhoki and Moll (2019). Those papers, however, focus on collateral constraints that prevent firms from operating at their optimal scale, which induces capital misallocation. A key insight from their analysis is that the degree of persistence in the stochastic idiosyncratic productivity process affects the magnitude of steady-state TFP losses from misallocation and the speed of transition towards steady state. Buera and Shin (2013) similarly focus on the effects of collateral constraints on misallocation, although in a model with decreasing returns to scale production and an occupational choice, with a focus on quantitative results. To clearly distinguish the new mechanism of risky production in our paper, we make labor the only factor of production and omit collateral constraints.

Our paper is also similar to Meh and Quadrini (2006) and Angeletos (2007) in that these papers study the effect of risky production, and not collateral constraints, on long-run allocations. Meh and Quadrini (2006) and Angeletos (2007), however, study a model with iid idiosyncratic risk focused on capital accumulation. Di Tella and Hall (2022) also model risky production in an environment with iid productivity shocks and an important role for investment, but in a model with aggregate risk focused on business cycle analysis. These models with iid shocks do not feature heterogeneous types and, thus, no potential for misallocation across types. We study a model without aggregate risk, capital, or capital constraints to focus on the long-run relationship between idiosyncratic risk, markups, and misallocation across persistently heterogeneous entrepreneurs.

2 The Model

2.1 Environment

Time is continuous and infinite, indexed by $t \geq 0$. There is a representative worker and a continuum of entrepreneurs, indexed by $i \in [0, 1]$, with preferences

$$U_w(c_w, \ell) = \mathbb{E}\left[\int_0^\infty e^{-\rho_w t}\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt\right], \tag{1}$$

$$U_e(c_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \log c_{it} dt\right]. \tag{2}$$

We assume entrepreneurs are more impatient than workers, $\rho_e > \rho_w > 0$ to obtain a stationary wealth distribution.

Each entrepreneur has a linear production technology that uses labor to produce the final good:

$$dY_{it} = z_{it}\ell_{it}dt + \sigma_y \ell_{it} dB_{yit},$$

$$dz_{it} = \mu_z(z_{it})dt + \sigma_z(z_{it})dB_{zit},$$

where B_{yit} and B_{zit} are two Brownian motions specific to the entrepreneur that have correlation λ_{yz}, and $\sigma_y, \sigma_z(z_{it}) > 0$. A central assumption in our setting is that production is risky. At time t when an entrepreneur chooses labor ℓ_{it}, he knows his expected marginal productivity z_{it}, but the final output is exposed to idiosyncratic risk through the term $\sigma_y \ell_{it} dB_{yit}$. An entrepreneur's marginal productivity z_{it} also evolves stochastically, following a diffusion driven by B_{zit}. We assume z_{it} remains in a bounded interval $[0, \bar{z}]$ and converges to a unique non-degenerate stationary distribution.

The resource constraints are[1]

$$c_t = \int_0^1 c_{it}di + c_{wt} = \int_0^1 z_{it}\ell_{it}di, \tag{3}$$

$$\ell_t = \int_0^1 \ell_{it}di. \tag{4}$$

2.2 Competitive Equilibrium

There are competitive markets for the consumption good, labor, the risk-free bond, and entrepreneurs' equity. The consumption good is the numeraire, with price normalized to 1. Markets are incomplete: entrepreneurs must retain a fraction $\phi \in (0,1)$ of their idiosyncratic risk. Budget constraints are

$$dn_{wt} = (r_t n_{wt} - c_{wt} + w_t\ell_t)dt, \tag{5}$$

$$dn_{it} = (r_t n_{it} - c_{it} + (z_{it} - w_t)\ell_{it})dt + (\phi\sigma_y)\ell_{it}dB_{yit}. \tag{6}$$

where n_{it} and n_{wt} are agents' net worth, r_t is the interest rate, and w_t the wage. Entrepreneurs' inability to share idiosyncratic risk is reflected in their budget constrain: they must retain an exposure ϕ to the idiosyncratic risk in their production.[2] Both entrepreneurs and workers face natural borrowing constraints. For entrepreneurs, this means $n_{it} \geq 0$. Workers cannot borrow more than the present value of labor income, $n_{wt} \geq -\int_t^\infty e^{-\int_t^s r_u du} w_s\ell_s ds$. Market clearing of the bond market requires

$$n_{wt} + \int_0^1 n_{it}di = 0. \tag{7}$$

[1] In equation (3) we are assuming a law of large numbers applies. That is, we focus on allocations such that $\int_0^1 \left(\int_0^t e^{-\int_0^s r_u du}\ell_{is}\sigma_y dB_{iys}\right) di = 0$.

[2] We can think of this as entrepreneurs selling a fraction $(1-\phi)$ of their equity to the other entrepreneurs and collecting a salary as CEO. See Appendix for different ways in which this can be implemented. Absent quantification of σ_y, it is without loss of generality to consider the $\phi = 1$ case, since ϕ and σ_y always appear together multiplicatively as $\phi\sigma_y$.

The *representative worker's problem* is to choose $(c_{wt} > 0, \ell_t \geq 0)$ to maximize his utility subject to his budget constraint and natural borrowing constraint, taking prices (r, w) and initial wealth n_{w0} given. An *entrepreneur's problem* is to choose $(c_{it} > 0, \ell_{it} \geq 0)$ to maximize his utility subject to his budget constraint and natural borrowing constraint, taking prices (r, w) and initial wealth n_{i0} given. Markets clear if equations 3 and 4 hold (with equation 7 then holding by Walras' Law).

For a given initial distribution of wealth n_{w0} and $n_{i0} > 0$ for $i \in [0, 1]$ satisfying (7), a *Competitive Equilibrium* is an allocation (c_w, ℓ_w) and (c_i, ℓ_i) for $i \in [0, 1]$, and market prices (r, w) such that the representative worker and each entrepreneur solve their problem and markets clear.

2.3 Characterizing the Competitive Equilibrium

Optimality Conditions. The representative worker's optimal allocation is characterized by an Euler equation and the first order condition for labor supply:

$$\frac{dc_{wt}}{c_{wt}} = (r_t - \rho_w)dt, \tag{8}$$

$$c_{wt}^{-1} w_t = \ell_t^{1/\psi}. \tag{9}$$

An entrepreneur's optimal allocation can be characterized by an Euler equation and the first order condition for labor demand:

$$\frac{dc_{it}}{c_{it}} = (r_t - \rho_e + \sigma_{cit}^2)dt + \sigma_{cit} dB_{yit}, \tag{10}$$

$$z_{it} - w_t \leq \sigma_{cit} \phi \sigma_y \quad (\text{ with equality if } z_{it} \geq w_t), \tag{11}$$

$$\sigma_{cit} = \frac{\ell_{it}}{n_{it}}(\phi \sigma_y) = \frac{\ell_{it}}{c_{it}}(\rho_e \phi \sigma_y), \tag{12}$$

Here we have used the fact that with log preferences $c_{it} = \rho_e n_{it}$ and there are no dynamic hedging motives. His consumption is therefore locally exposed only to the idiosyncratic risk in his output B_{yt} (and not the idiosyncratic risk in his productivity B_{zt}, although they may be correlated). The term σ_{cit}^2 in the Euler equation captures the precautionary saving motive. The demand for labor trades off the expected marginal profit from hiring labor, $z_{it} - w_t$, against the increase in the exposure to risk. The $\sigma_{cit}\phi\sigma_y$ term is a risk premium on labor: it captures the covariation between the marginal utility of the entrepreneur and the fraction ϕ of the marginal product of labor that cannot be shared. Finally, the last term uses the budget constraint to express the exposure to risk on the entrepreneur's consumption in terms of his labor demand ℓ_{it}, and uses $c_{it} = \rho_e n_{it}$ to eliminate the net worth.

Entrepreneurs hire less labor compared to the complete markets case. Hiring labor is like investing in a risky asset and they consider the trade off between the expected return and the risk associated with hiring. Even though production of the homogeneous good is linear, the most productive entrepreneur does not hire all the labor because that would expose him to too much risk.

The risk premium on labor creates labor wedges at the entrepreneur level that look like markups that are increasing in productivity z_{it}. Rearranging equation 11 shows:

$$1 + \pi_{it} := \frac{z_{it}}{w_{it}}. \tag{13}$$

That is the price of the good is higher than the unit labor cost for all entrepreneurs with $z_{it} > w_t$. But risk markups capture the marginal cost of exposing the entrepreneur to risk. The risk-adjusted marginal product of labor, $z_{it} - \sigma_{cit}\phi\sigma_y$, is equalized to the labor cost w_t for all entrepreneurs who have positive labor demand, $\ell_{it} > 0$. Thus, there is no true markup in the sense that the risk adjusted marginal cost is equal to the price of the good for all active entrepreneurs. At the aggregate level, however, risk markups show up as a labor wedge generating misallocation that reduces TFP. Let Z_t be aggregate TFP and $1 + \pi_t$ the aggregate labor wedge

$$Z_t := \frac{c_t}{\ell_t} = \int_0^1 z_{it} \frac{\ell_{it}}{\ell_t} di,$$
$$1 + \pi_t := \frac{Z_t}{w_t} = \int_0^1 \frac{z_{it}}{w_t} \frac{w_t \ell_{it}}{w_t \ell_t} di.$$

Aggregate productivity Z_t is the labor-weighted average productivity and the aggregate labor wedge $1 + \pi_t$ is the labor-weighted average labor wedge.

Aggregation and a Separation Property. The environment with log preferences and linear production allows for easy aggregation and a powerful separation result. We can solve for the cross-sectional allocations separately from aggregate employment and output. Because of the homothetic preferences and linear budget constraints, when computing equilibrium allocations we do not need to keep track of the joint distribution of wealth n_{it} and productivity z_{it}. It is enough to keep track of the aggregate wealth (or equivalently, consumption) of entrepreneurs of each level of productivity. Let η_t represent the distribution of consumption at time t, in shares of aggregate output:

$$\eta_{it} = c_{it}/c_t, \tag{14}$$

$$\eta_{wt} = c_{wt}/c_t. \tag{15}$$

Let $\eta_{zt}(z) = \int_{\{z_{it}=z\}} \eta_{it} di$ be the consumption share of entrepreneurs with productivity z and $\eta_{et} = \int_0^\infty \eta_{zt}(z) dz = \int_0^1 c_{it} di = 1 - \eta_{wt}$ be the consumption share of entrepreneurs.

We can rewrite the entrepreneurs' optimality conditions as

$$\sigma_{cit} = \frac{(z_{it} - w_t)^+}{\phi \sigma_y}, \tag{16}$$

$$\ell_{it} = \sigma_{cit} \eta_{it} \frac{c_t}{\rho_e \phi \sigma_y}. \tag{17}$$

Use the resource constraints to obtain expressions for the wage and aggregate productivity as function of η_{zt}:

$$\frac{1}{\rho_e \phi \sigma_y} \times \int_{w_t}^{\bar{z}} z \eta_{zt}(z) \frac{z - w_t(\eta_{zt})}{\phi \sigma_y} dz = 1, \tag{18}$$

$$\left(\frac{1}{\rho_e \phi \sigma_y} \times \int_{w_t}^{\bar{z}} \eta_{zt}(z) \frac{z - w_t(\eta_{zt})}{\phi \sigma_y} dz \right)^{-1} = \frac{c_t}{\ell_t} = Z_t(\eta_{zt}), \tag{19}$$

as well as the exposure to risk of each productivity type z

$$\sigma_{cit} = \sigma_{ct}(z_{it}; \eta_{zt}) = \frac{(z_{it} - w_t(\eta_{zt}))^+}{\phi \sigma_y}. \tag{20}$$

We can use the goods resource constraint and the Euler equations for the worker and entrepreneurs to describe the evolution of $\eta_{zt}(z)$ by a Kolmogorov forward equation (KFE) that only depends on η_{zt} itself,

$$\begin{aligned}
&\partial \eta_z(z,t) = \\
&\quad \eta_z(z,t) \left[-\mu_z'(z) + \mu_\eta(z, \eta_z(\cdot,t)) + \sigma_z'(z)^2 + \sigma_z(z)\sigma_z''(z) - [\sigma_z'(z)\sigma_c(z,\eta_z(\cdot,t)) + \sigma_z(z)\sigma_c'(z,\eta_z(\cdot,t))] \left(\lambda_{yz} + \sqrt{1-\lambda_{yz}^2} \right) \right] \\
&\quad + \eta_z'(z,t) \left[-\mu_z(z) + 2\sigma_z(z)\sigma_z'(z) - \sigma_z(z)\sigma_c(z,\eta_z(\cdot,t)) \left(\lambda_{yz} + \sqrt{1-\lambda_{yz}^2} \right) \right] \\
&\quad + \eta_z''(z,t) \left[\frac{1}{2} \sigma_z^2(z) \right],
\end{aligned}$$

where $\mu_z(z)$, $\sigma_z(z)$, and λ_{yz} are primitives, and $\sigma_c(z, \eta_z(\cdot,t))$ and $\mu_\eta(z, \eta_z(\cdot,t))$ are equilibrium objects given by:

$$\begin{aligned}
\sigma_c(z, \eta_z(\cdot,t)) &= \frac{(z_t - w_t)^+}{\phi \sigma_y}, \\
\mu_\eta(z, \eta_z(\cdot,t)) &= \bar{\rho}_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2 \sigma_y^2} - \int \eta_{zt} \frac{(z_t - w_t)^2}{\phi^2 \sigma_y^2} dz,
\end{aligned}$$

with w_t a function of η_{zt}. See Appendix F for details of the KFE derivation.

Constructing the Equilibrium. This separation result allows us to compute an equilibrium in two steps. In step one, for a given n_{w0} and n_{i0}, use $c_{i0} = \rho_e n_{i0}$ to compute initial η_{i0} and, thus, $\eta_{z0}(z)$ and η_{e0}. Solve the KFE forward to obtain the path of $\eta_{zt}(z)$ and, thus, η_{et}. Then use (18)-(20) to solve for the path of w_t, Z_t, and $\sigma_{ct}(z)$. At this point we must verify that there exists a positive w_t that solves (18) using $\eta_{zt}(z)$ at all t.[3]

Due to the separation result, step one does not involve aggregate output c_t, aggregate employment ℓ_t, or interest rates r_t. In step two, we determine these by using the representative worker's labor supply condition to solve for aggregate employment and the definition of aggregate TFP to compute aggregate output

$$\begin{aligned} (\overbrace{(1-\eta_{et})Z_t\ell_t}^{c_{wt}})^{-1} w_t &= \ell_t^{1/\psi}, \\ c_t &= Z_t \ell_t. \end{aligned}$$

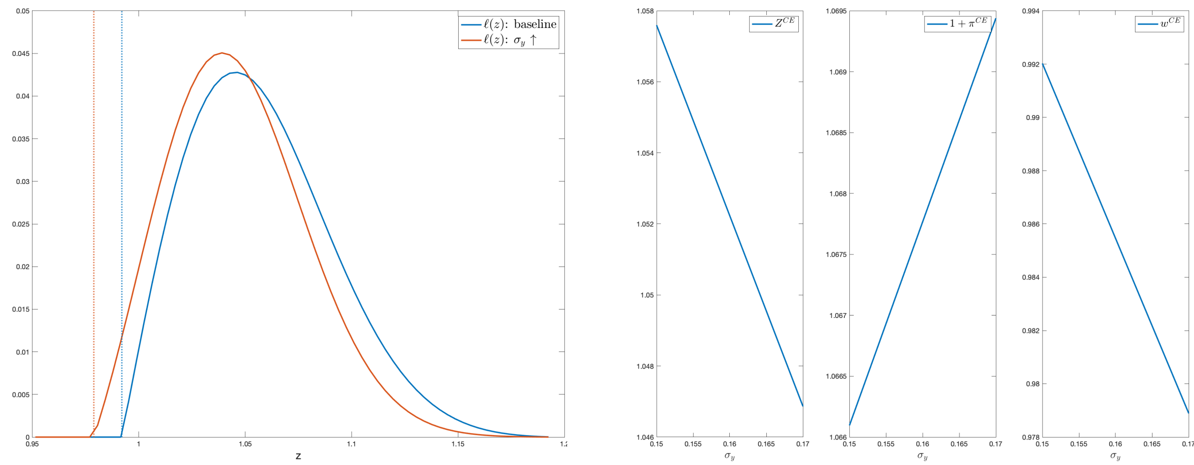
Once we have the path of ℓ and c we can construct the rest of the equilibrium. The representative worker's consumption is $c_{wt} = (1-\eta_{et})c_t$, and their Euler equation determines the interest rate

$$r_t = \rho_w + \frac{\dot{c}_{wt}}{c_{wt}}.$$

Set $c_{i0} = \eta_{i0} c_0$. Since $\sigma_{cit} = \sigma_{ct}(z_{it}; \eta_{zt})$ from equation (20), we can use entrepreneurs' Euler equation (10) to determine their consumption c_{it} and then their labor demand FOC in equation (12) determines ℓ_{it} . We set the net worth of each entrepreneur to $n_{it} = c_{it}/\rho_e$ and the net worth of the worker to $n_{wt} = -\int_0^1 n_{it} di$.

Proposition 1. *Let $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ be as constructed by the above procedure with associated initial wealth n_{w0} and n_{i0}. Assume that for all t, $w_t, c_{it}, c_{wt} > 0$ and $\lim_{T\to\infty} e^{-\int_0^T r_u du} n_{wT} = 0$. Then $\mathcal{C}$ is a competitive equilibrium for initial wealth n_{w0} and n_{i0}.*

Proof. See Appendix C.



(a) Total Labor Hired by type-z Entrepreneurs (b) TFP, Aggregate Markup, and Wage

Figure 1: Comparative Static w.r.t. σ_y

2.4 A Numerical Example

To illustrate key characteristics of the competitive equilibrium, we explore comparative statics around some roughly chosen baseline parameter values. See Appendix G for details. Figure 1 explores how labor demand, TFP, and the aggregate markup change when hiring labor becomes more risky (increase in σ_y).

Figure 1a plots the distribution of total labor hired by entrepreneurs of type z. The blue distribution is for the baseline calibration and the orange distribution is for a counterfactual equilibrium with a larger σ_y. The vertical dotted lines are the wages associated with each equilibrium. First, we see zero labor is hired by entrepreneurs with conditional expected productivity lower than the wage. There is a steady state distribution of productivity z, given by the exogenous stationary persistent stochastic process, that is log-normally distributed around a mean of 1. Three forces determine the distribution of labor across productivity types. First, the density of is declining in productivity for $z > 1$. Second, the entrepreneur's optimal labor hiring policy is linearly increasing in productivity. Third, labor demand is a function of the wealth distribution.

[3] If not, no competitive equilibrium exists that is consistent with initial condition n_{i0}, n_{w0}.

In this calibration, there is not much labor hired by low productivity types close to the wage threshold even though there are many entrepreneurs of this type because they each hire little labor. Total labor hired increases as z increases even though the density of entrepreneurs is decreasing, because each entrepreneur is hiring more workers. Finally, as z increases, the density of entrepreneurs falls fast enough that the total labor hired by z-type entrepreneurs falls even as each entrepreneur continues to hire more workers.

In the equilibrium with more risk (larger σ_y), each entrepreneur would hire less labor holding the wage constant. In equilibrium, the wage falls, bringing previously inactive entrepreneurs into production. The end result is that the largest producers reduce labor demand and the smallest producers increase labor demand. Figure 1b shows the effect of an change in σ_y on key macro aggregates. As σ_y increases, the wage falls. TFP also falls in response, because labor is shifting away from the most productive producers towards the least productive producers. The aggregate markup rises, reflecting the increase in the risk premia. In this model risk generates misallocation, lowering TFP. We know turn to analyzing if the allocation is constrained efficient, or if a constrained efficient planner who cannot complete the market can nonetheless improve efficiency.

3 Optimal Policy

Consider a social planner who has access to two policy instruments: (1) a uniform labor tax τ on the representative worker rebated lump-sum $T_t = \tau \ell_t$, and (2) lump-sum transfers across agents at time $t = 0$. We are interested in studying the efficiency properties of the competitive equilibrium. The $t = 0$ transfers can accomplish any redistribution the social planner may desire, so we can focus on understanding which interventions improve efficiency.[4]

In principle we could study a planner who has access to entrepreneur-specific labor taxes and interest rate taxes. In the absence of a mechanism design approach that delivers the

[4] A uniform interest rate subsidy is a redundant policy instrument as long as the planner has access to $t = 0$ lump-sum transfers. All agents have the same interest rate in their Euler equations, and a subsidy/tax that affects the return on savings equally for all agents simply serves to redistribute without affecting efficiency.

policy instruments needed to implement the planner's optimal allocation, we need to carefully select the tools we give to the planner to avoid studying an uninteresting problem. Motivated by such asymmetric information concerns, we think it may not be feasible for a planner to successfully use entrepreneur-specific taxes without inducing unobserved changes in entrepreneur behavior. Thus, we think a uniform tax is the most interesting case. In order to better understand how to optimally use a uniform labor tax, we will consider a case in which the planner also has access to a tax on entrepreneur saving in Section 3.2.

3.1 The Planner's Problem

A labor tax rebated to workers distorts the labor margin without affecting any budget constraint in equilibrium. The representative worker's budget constraint is

$$dn_{wt} = (n_{wt}r_t - c_{wt} + w_t(1-\tau_t)\ell_t + T_t)dt, \tag{21}$$

with natural borrowing limit $n_{wt} \geq -\int_0^\infty e^{-\int_0^s r_u du}(w_s(1-\tau_s)\ell_s + T_s)ds$. In equilibrium $T_t = \tau_t \ell_t$. The equilibrium condition for labor supply becomes

$$\begin{aligned} \ell_t^{1/\psi} &= c_{wt}^{-1} w_t (1-\tau_t), \\ &= ((1-\eta_{et})Z_t\ell_t)^{-1} w_t(1-\tau_t). \end{aligned} \tag{22}$$

All other equilibrium conditions are unchanged. The separation property of the competitive equilibrium implies that by choosing the labor tax, the planner can choose aggregate labor and output without affecting other aspects of the allocation. The separation result of the competitive equilibrium implies that the planner cannot reduce misallocation with the labor tax and can only affect misallocation by choosing the initial distribution of wealth.

The *representative worker's problem* is to maximize utility (1) subject to dynamic budget constraint (21). The *entrepreneur's problem* is unchanged. A *competitive equilibrium* *with uniform labor tax and lump-sum transfers* is an allocation (c_w, ℓ, c_i, ℓ_i), prices (r, w), taxes (τ, T) and initial wealth (n_{i0}, n_{w0}) such that (a) the representative worker and each entrepreneur solve their problem given their initial wealth, (b) markets clear, and (c) $T_t = \tau_t \ell_t$ and $\int n_{i0} di + n_{w0} = 0$. An allocation (c_w, ℓ, c_i, ℓ_i) is *implementable* if there exist prices (r, w), taxes (τ, T) and initial wealth (n_{w0}, n_{i0}) such that taken together they constitute a *competitive equilibrium with uniform labor tax and lump-sum transfers.* An implementable allocation (c_w, ℓ, c_i, ℓ_i) is *optimal* if it maximizes

$$\mathbb{E}\left[\Gamma_w \int_0^\infty e^{-\rho_w t}\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt + \Gamma_e \int_0^1 \left(\int_0^\infty e^{-\rho_e t} \log c_{it} dt\right) di\right], \tag{23}$$

where Γ_w and Γ_e are the Pareto weights on the representative worker and entrepreneurs. For simplicity of exposition, we assume that the planner does not care about entrepreneurs individually ($i.e.$, $\Gamma_i = \Gamma_e \ \forall\ i$).

Transforming the planner's problem. We can make the planner's problem more manageable taking advantage of the separation property of the competitive equilibrium. First, any implementable allocation must satisfy the resource constraints (3) and (4), the Euler equations (8) and (9), and the first order conditions for exposure to risk (20) and labor demand (17). Given these constraints, once the planner chooses the initial distribution η_{z0}, η_{zt} will then follow its KFE and pin down $Z(\eta_{zt})$, $w(\eta_{zt})$ and $\sigma_c(z; \eta_{zt})$ according to (18)-(20). It can then choose a path for aggregate labor ℓ without affecting these equilibrium objects. We can therefore recast the planner's problem as choosing an initial distribution η_{z0} and a path for aggregate labor ℓ to maximize

$$\int_0^\infty e^{-\rho_w t}\Gamma_w \left(\log c_{wt}\left(\frac{\Gamma_w + \Gamma_e e^{-(\rho_e - \rho_w)t}}{\Gamma_w}\right) - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt \tag{24}$$
$$+ \int_0^1 \int_0^\infty \Gamma_e e^{-\rho_e t} \frac{1}{\rho_e} \frac{1}{2} \sigma_c(z_{it}; \eta_{zt})^2 dt di + \Gamma_e \frac{\int_0^1 \log \eta_{i0} di - \log \eta_{w0}}{\rho_e},$$

subject to

$$c_{wt} = (1 - \eta_{et}) Z(\eta_{zt}) \ell_t$$

and the KFE for η_{zt}. See Appendix D for details of the derivation.

The advantage of this formulation is that the first term is a standard optimization problem that does not involve any cross-sectional objects. The second term captures the fact that, given the path of consumption for the representative worker c_{wt}, if the entrepreneurs are exposed to risk $\sigma_c(z_{it}; \eta_{zt})$, their consumption must be growing faster to compensate them (precautionary savings) but they dislike the exposure to risk (risk aversion). It depends on the choice of η_{z0} through the evolution of the distribution η_{zt}, but it is not affected by the choice of ℓ. The third term captures how Pareto weights distort initial consumption shares: if entrepreneurs' Pareto weight is higher, the planner will give them a larger share of aggregate consumption at time $t = 0$, other things equal.

We can solve the problem in two steps. First, given η_{zt} we can choose ℓ_t to maximize the first term. The first order condition for ℓ_t is

$$((1 - \eta_{et}) Z(\eta_{zt}) \ell_t)^{-1} \times Z(\eta_{zt}) \times \frac{1 - \eta_{et}}{1 - \frac{e^{-(\rho_e - \rho_w)t} \Gamma_e}{\Gamma_w + e^{-(\rho_e - \rho_w)t} \Gamma_e}} = \ell_t^{1/\psi}. \tag{25}$$

In the second step, we can then plug this expression back into the objective function and optimize over η_{z0}. This second step is a relatively simple optimization over a single distribution that can be carried out numerically.

Constructing the optimal allocation. We can construct the allocation following the same steps we used to construct the competitive equilibrium, except that we do not need to use the first order condition for labor supply to determine labor supply ℓ_t. Instead, we know ℓ_t and we need to find the labor tax τ_t and the initial wealth distribution, n_{w0} and n_{i0}, that implement the allocation. Comparing the planner's FOC in equation (25) with the equilibrium labor supply condition with a labor tax in equation (22), the tax that supports the planner's allocation is

$$1 - \tau_t = (1 + \pi_t) \times \frac{1 - \eta_{et}}{1 - \frac{e^{-(\rho_e - \rho_w)t}\Gamma_e}{\Gamma_w + e^{-(\rho_e - \rho_w)t}\Gamma_e}}. \tag{26}$$

We then set the representative worker's consumption to $c_{wt} = (1 - \eta_{et})Z_t \ell_t$, and use his Euler equation (8) to determine the interest rate. Set $c_{i0} = \eta_{i0} c_0$ and use entrepreneurs' Euler equation (10) and $\sigma_{cit} = \sigma_{ct}(z_{it})$ to pin down their consumption c_{it}, and set $\ell_{it} = \sigma_{ct}(z_{it}) \frac{c_{it}}{\rho_e \phi \sigma_y}$. The tax rebate is $T_t = \tau_t w_t \ell_t$ and the net worth are $n_{it} = c_{it}/\rho_e$ and $n_{wt} = -\int_0^1 n_{it} di$.

Proposition 2. *Let $\mathcal{S} = (c_w, \ell, c_i, \ell_i)$ be an allocation built as described above, with associated prices (r, w), taxes (τ, T) and initial wealth (n_{i0}, n_{w0}). Assume that for all t, $w_t(1 - \tau_t), c_{it}, c_{wt} > 0$ and $\lim_{T \to \infty} e^{-\int_0^T r_u du} n_{wT} = 0$. Then $\mathcal{S}$ is an optimal allocation.*

Proof. See Appendix E.

Long-run Optimal Tax. Assume that in the long run as $t \to \infty$ the consumption distribution converges to a stationary distribution η_z^{ss} and the corresponding $\eta_{et} \to \eta_e^{ss} \in (0, 1)$. Note, if the stationary distribution is independent of initial conditions, then the separation result implies that the planner cannot affect misallocation in the long run. Furthermore, since entrepreneurs are more impatient than workers, $\rho_e > \rho_w$, their Pareto weight converges to 0. As a result, the optimal tax converges to

$$1 - \tau^{LR} = (1 + \pi^{ss}) \times (1 - \eta_e^{ss}). \tag{27}$$

We can sign the optimal tax in the long run in terms of consumption shares as

$$1 - \tau^{LR} = \frac{c_w^{ss}}{w^{ss} \ell^{ss}}. \tag{28}$$

If in the long run entrepreneurs consume their profits and workers consumer their labor income, then the optimal tax is zero. If instead workers consume less than their labor income (because they pay interest on debt to entrepreneurs), then the optimal tax $\tau^{LR} > 0$. As can be seen in the planner's labor FOC (equation 25), in the long run the planner sets $\ell^{ss} = 1$. Since $\eta_e^{ss} \in (0, 1)$, $n_w^{ss} < 0$, so by the budget constraint of the worker $c_w^{ss} < w^{ss}\ell^{ss}$. In the long run, workers are paying interest on their debt to entrepreneurs, so workers consume less than their labor income and the optimal labor tax is positive.

The optimal labor tax can be understood in terms of an inefficient risk premium that pushes for a labor subsidy (negative tax) and an inefficient precautionary saving motive that pushes for a labor tax. In the long-run, the precautionary inefficiency dominates.

The key insight of this paper is that the single feature of uninsurable idiosyncratic risk generates both inefficient risk markups and inefficient precautionary saving and that the single policy instrument of a labor tax cannot simultaneously correct both inefficiencies. As a result, optimal labor tax policy when misallocation is caused by uninsurable risk is the opposite of what would be optimal when misallocation is caused by monopolistic competition (in the Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023) sense). In the next subsection we clarify this insight.

3.2 Understanding the Inefficiency

The optimal labor tax in equation (26) can be understood in terms of two elements: inefficient risk premium and inefficient precautionary saving. The first term on the right is the aggregate markup (labor wedge), $1 + \pi_t$. The optimal labor tax aims to undo the average labor wedge with a subsidy, which would be the optimal policy if the wedge was created by monopolistic competition. Here, however, the wedge represents a risk premium that captures the cost of exposing the entrepreneur to more risk. Why is it inefficient? There is an externality associated with risk sharing. To see this, we can write entrepreneur's exposure to risk in the following way:

$$\sigma_{cit} = (\rho_e \phi \sigma_y) \frac{\ell_{it}}{c_{it}} = (\rho_e \phi \sigma_y) \left(\frac{1}{\eta_{it} Z_t(\eta_{zt})} \right) \frac{\ell_{it}}{\ell_t}. \tag{29}$$

If an entrepreneur increases his labor demand ℓ_{it} individually, he is taking more risk. This is why in the competitive equilibrium entrepreneurs demand a risk premium as compensation. But the planner can consider a deviation that raises all entrepreneur ℓ_{it} at the same time and in the same proportion, so that the numerator ℓ_{it} and the denominator ℓ_t both increase proportionally and each entrepreneur's exposure to risk is unchanged. In terms of equilibrium objects, $\sigma_{cit} = (\phi \sigma_y) \frac{\ell_{it}}{n_{it}}$. An individual entrepreneur takes n_{it} as given when choosing ℓ_{it}. If the planner raises aggregate output c_t at time t by raising ℓ_t, then $n_{it} = c_{it}/\rho_e$ raises proportionally; this must occur because $\eta_{it} = c_{it}/c_t$ is predetermined. Thus, σ_{cit} is unchanged. In the background, market prices are adjusting to deliver the appropriate change in net worth. Entrepreneurs enter period t with higher net worth because the past rate of return on their savings (the interest rate) increases with this deviation. So the risk premium is correctly capturing the private cost of exposing entrepreneurs to risk at the margin, but not the correct social cost.

The second term in equation (26) is

$$\frac{1 - \eta_{et}}{1 - \frac{e^{-(\rho_e - \rho_w)t} \Gamma_e}{\Gamma_w + e^{-(\rho_e - \rho_w)t} \Gamma_e}}$$

and captures an inefficient precautionary saving motive. The numerator is the share of consumption that goes to the representative worker and the denominator is their relative Pareto weight. In the absence of incomplete risk sharing, this ratio would always be one. The planner would give a share of consumption to the worker reflecting their share of Pareto weights, taking into account the different impatience rates. With incomplete risk sharing, this is not the case. Entrepreneurs must be exposed to risk if they are to employ labor, and the exposure to risk will induce them to save to self insure (precautionary saving). This will lead to an inefficiently high consumption share for entrepreneurs, in an analogous way to how precautionary saving leads to excessive capital accumulation in an Aiyagari model with incomplete markets. The difference is that here we don't have capital that entrepreneurs can accumulate. Instead, they excessively accumulate claims on the representative worker who is not exposed to uninsurable risk (and therefore does not have a precautionary motive).

But even if entrepreneurs' share of consumption is too high, why does this show up as an inefficiency in the aggregate level of labor? The reason is that, from the perspective of the planner, if he increases labor and output at time t, the share of the extra output that goes to entrepreneurs above their Pareto weights is wasted. So he acts as if the marginal product of labor was not Z_t but rather Z_t times this ratio (as can be seen clearly in the first order condition in equation (25).

A thought experiment to understand the dynamic inefficiency. To understand the dynamic inefficiency better, consider what the planner would do if he could also tax entrepreneurs' savings. To simplify the thought experiment, assume the planner doesn't value the entrepreneurs' utility, $\Gamma_e = 0$. Also, assume for now that there is only one productivity, $z_{it} = \bar{z}$ always; there are no persistent productivity shocks.

Under these assumptions, aggregate productivity is $Z_t = \bar{z}$, and to compute an equilibrium we only need to keep track of the aggregate consumption share of entrepreneurs, η_{et}, instead of the whole distribution η_{zt}. Furthermore, the entrepreneurs' saving tax τ_{st} allows the planner to control the evolution of η_{et}, while the labor tax $\tau_{\ell t}$ allows him to control ℓ_t. The wage is given by

$$\frac{1}{\rho_e \phi \sigma_y} \bar{z} \eta_{et} \frac{\bar{z} - w_t(\eta_{et})}{\phi \sigma_y} = 1, \tag{30}$$

and the labor tax is set so the worker's first order condition holds, $((1-\eta_{et})\bar{z}\ell_t)^{-1} w_t (1-\tau_{\ell t}) = \ell_t^{1/\psi}$. Entrepreneurs' idiosyncratic risk is given by $\sigma_{ct} = \sigma_c(\eta_{et}) = \frac{(\bar{z} - w(\eta_{et}))^+}{\phi \sigma_y}$, and the KFE simplifies to an ODE

$$\frac{d\eta_{et}}{\eta_{et}} = (1-\eta_{et})(\rho_w - \rho_e + \sigma_{ct}^2 - \tau_{st}). \tag{31}$$

Before solving for the optimal plan with these two instruments, let's revisit our previous result when the planner only has a labor tax, but now in this simplified environment. With $\Gamma_e = 0$, the optimal labor supply is given by equation (25), so $\ell^{SP} = 1$. It coincides with the first best and is constant because of balanced growth preferences. The labor tax trades off the inefficient risk premium on labor and the inefficient precautionary savings,

$$1 - \tau_\ell^{SP} = (1+\pi_t) \times (1-\eta_{et}).$$

The planner can choose the initial η_{e0} but after that η_{et} evolves according to its KFE. As long as it converges to a steady state in the long-run, we know that the optimal tax is positive (a tax), $\tau_\ell^{LR} > 0$. This demonstrates that the simplified environment of $\Gamma_e = 0$ and $z_{it} = \bar{z}$ did not eliminate the features of the optimal policy that we are trying to explain.

Now consider the planner who also has a tax on saving for entrepreneurs. The planner can now control η_{et} using the saving tax τ_{st}. Since the planner has zero Pareto weight on entrepreneurs, he would like to give them zero consumption. But at zero consumption they wouldn't be able to produce at all, because choosing $\ell_{it} > 0$ would surely violate the natural borrowing constraint. Thus, we will consider a sequence of allocations with a small but positive initial η_{e0} and

$$\eta_{et} = \frac{e^{-(\rho_e-\rho_w)t}\eta_{e0}}{1-\eta_{e0}+e^{-(\rho_e-\rho_w)t}\eta_{e0}}. \tag{32}$$

Then we let $\eta_{e0} \to 0$, so $\eta_{et} \to 0$ uniformly and the planner's objective is

$$\max_{c_w,\ell} \int_0^\infty e^{-\rho_w t}\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt \tag{33}$$

subject to $c_{wt} = \bar{z}\ell_t$. This is in fact the first best problem corresponding to $\Gamma_e = 0$. The first order condition for labor is $\ell_t^{1/\psi} = c_{wt}^{-1}\bar{z} = \ell_t^{-1}$, which implies $\ell_t^{FB} = 1$ and $c_w^{FB} = \bar{z}$.

Now let's see how these sequences of allocations are implemented. From equation (30) we see that as $\eta_{et} \to 0$, the wage paid by entrepreneurs must become unboundedly negative, $w(\eta_{et}) \to -\infty$. Entrepreneurs are very poor and yet are asked to take risk associated with the constant first-best level of employment, $\sigma_c(\eta_{et}) \to \infty$, so they demand a very large discount on labor to compensate for this risk, $\bar{z} - w(\eta_{et}) \to \infty$. With negative w_t, to get the worker to supply labor the planner must introduce a labor subsidy,

$$(1 - \tau_{\ell t})w_t = \bar{z}, \tag{34}$$

so that the after-tax wage is equal to the marginal product of labor $\bar{z}$ (while $w_t < 0$, the subsidy per hour worked is $\tau_{\ell t} w_t \to \infty$). Finally, the saving tax exactly counteracts the precautionary saving motive,

$$\tau_{st} = \sigma_{ct}^2 \to \infty. \tag{35}$$

In summary, granting the planner a tax on entrepreneurs' saving allows him to implement the first best. He expropriates entrepreneurs, and since they are poor but must still have the first-best level of employment, they demand a large risk premium on labor and have a large precautionary saving motive. The planner uses a labor subsidy to undo the effect of the risk premium on labor and a tax on entrepreneurs' saving to eliminate their precautionary saving. Without access to the tax on saving, the planner cannot undo entrepreneurs' precautionary saving and must allow them to accumulate a larger consumption share over time. This dynamic inefficiency distorts the labor margin. In the long-run, precautionary saving turns a labor subsidy into a labor tax.

3.3 Transition Dynamics.

Figure 2: Transition Dynamics

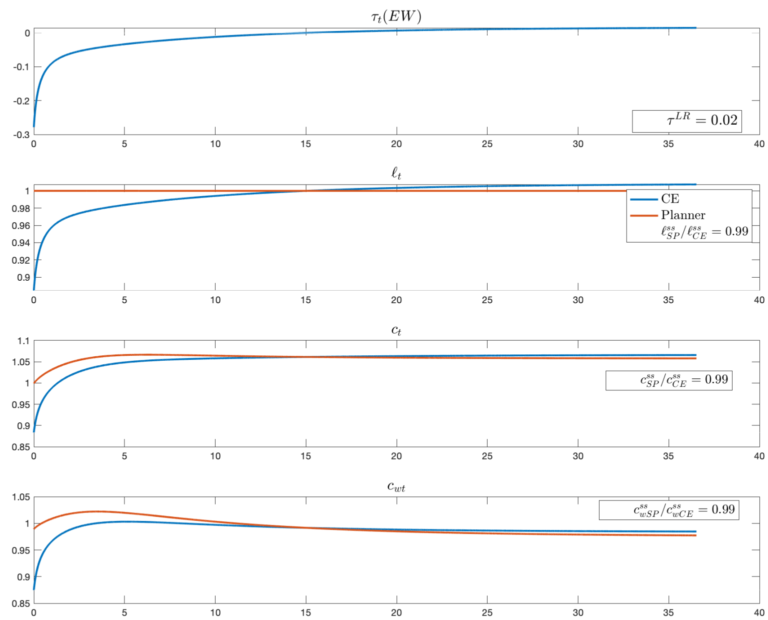


Figure 2 plots the transition dynamics of the planner's allocation compared to the competitive equilibrium. For ease of exposition, we consider the case where the Pareto weight on entrepreneurs is zero and all entrepreneurs start with the same productivity $z_{i0} = 1$ and with degenerate initial condition $\eta_{zt}(z) = \delta(z; z_0)\eta_e$ for $z_0 = 1$. We can therefore recast the planner's problem as choosing an initial single scalar $\eta_{e0} \in (0, 1)$ and a path for aggregate labor ℓ.

The planner implements a constant labor supply, while the equilibrium labor supply is increasing over time. Initially workers have too much wealth, so they don't want to work enough. The planner subsidizes their wages to increase the labor supply. This increase in output leads to increased consumption in aggregate. The entrepreneurs save a larger fraction of their income than workers, and over time accumulate wealth, inducing financial transfers from workers to entrepreneurs that grow over time on the transition. Eventually the workers accumulate so much wealth that the planner switches from subsidizing labor to taxing labor, to avoid workers working too hard and sending a large financial transfer to entrepreneurs in the form of interest on debt. In our very rough calibration, long-run taxes are small (2 percent). While we do not have data on the consumption share of workers, to the degree the consumption share is close to the income share, the tax will be small. A planner who mistakenly believed that measured markups reflected market power and, thus, levied a large labor subsidy would be far from implementing optimal policy in our environment.

4 Conclusion

This paper develops a theoretical framework to study how uninsurable idiosyncratic risk affects misallocation. Hiring labor is like investing in a risky asset, so entrepreneurs trade off the expected profits against the uninsurable risk and demand a risk premium on labor. This risk premium creates labor wedges at the entrepreneur level that look like markups that are increasing in productivity. However, the risk-adjusted marginal product is equalized for all active entrepreneurs. Thus, there is no true markup in the sense that the risk adjusted marginal cost is equal to the price of the good for all active entrepreneurs. At the aggregate level, however, risk markups show up as a labor wedge generating misallocation that reduces TFP.

We show how a planner who cannot complete markets, but can use taxes, subsidies, and transfers, can improve welfare. The optimal labor tax can be understood in terms of an inefficient risk premium that pushes for a labor subsidy (negative tax) and an inefficient precautionary saving motive that pushes for a labor tax. In the long-run, the precautionary inefficiency dominates and the optimal policy is a tax, which is the opposite of the optimal subsidy typically found in the literature with alternative microfoundations for measured markups. Since policy recommendations differ qualitatively, further research to identify the relative importance of uninsurable risk and monopoly power in generating misallocation is highly valuable.

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Appendix

A Entrepreneur Optimality

The entrepreneurs' problem is to chose non-negative adapted processes (c_i, ℓ_i) to maximize utility

$$U(c_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \log c_{it} dt\right] \tag{36}$$

subject to the dynamic budget constraint

$$dn_{it} = (r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t))dt + \ell_{it}\sigma_y \phi dB_{yit} \tag{37}$$

with initial wealth n_{i0} given and the natural borrowing limit $n_{it} \geq 0$.

Any plan (c_i, ℓ_i) satisfying the dynamic budget constraint satisfies the intertemporal budget constraint

$$\mathbb{E}\left[\int_0^\infty \xi_{it} c_{it} dt\right] \leq n_{i0}, \tag{38}$$

where ξ_{it} is the solution to

$$d\xi_{it}/\xi_{it} = -r_t dt - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} dB_{yit}, \tag{39}$$

with $\xi_{i0} = 1$.

To see this, use Itô's lemma to compute

$$\begin{aligned} d(\xi_{it} n_{it}) &= \xi_{it}(r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t) - r_t n_{it} - \frac{(z_{it} - w_t)^+}{\phi\sigma_y}\ell_{it}\sigma_y\phi)dt + \xi_{it}\left(\ell_{it}\sigma_y\phi - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} n_{it}\right) dB_{yt}, \\ &= \xi_{it}(-c_{it} + \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right))dt + \xi_{it}\left(\ell_{it}\sigma_y\phi - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} n_{it}\right) dB_{yt}. \end{aligned}$$

Integrate and take expectations to obtain

$$\mathbb{E}\left[\xi_{i\tau_j} n_{i\tau_j}\right] = \xi_{i0} n_{i0} + \mathbb{E}\left[\int_0^{\tau_j} \xi_{it}(-c_{it} + \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right))dt\right],$$

where $\{\tau_j\}$ is an increasing sequence of stopping times with $\lim \tau_j = \infty$ a.s.. Reorganizing we get

$$\mathbb{E}\left[\int_0^{\tau_j} \xi_{it}(c_{it} - \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right))dt\right] + \mathbb{E}\left[\xi_{i\tau_j} n_{i\tau_j}\right] = \xi_{i0} n_{i0}.$$

Notice that $\xi_{it}(c_{it} - \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right)) \geq \xi_{it} c_{it} \geq 0$. Take limits as $j \to \infty$ using the monotone convergence theorem and recall $\xi_{it} n_{it} \geq 0$ for any plan that satisfies the natural borrowing constraint. We obtain

$$\mathbb{E}\left[\int_0^{\infty} \xi_{it} c_{it} dt\right] \leq n_{i0}.$$

We will therefore consider a relaxed problem: choose consumption c_i to maximize utility $U(c_i)$ subject to the intertemporal budget constraint (38) and then later check that the proposed solution satisfies the dynamic budget constraint. We have a first-order condition

$$e^{-\rho_e t} c_{it}^{-1} = \lambda \xi_{it}, \tag{40}$$

where the value of $\lambda > 0$ ensures the constraint holds with equality:

$$\mathbb{E}\left[\int_0^{\infty} \xi_{it} c_{it} dt\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\rho_e t} \lambda^{-1} dt\right] = \frac{1}{\rho_e \lambda} = n_{i0},$$
$$\implies \lambda = \frac{1}{\rho_e} \frac{1}{n_{i0}}.$$

These conditions are sufficient for optimality of the relaxed problem. To see this, consider any alternative plan $\tilde{c}_i$ satisfying the intertemporal budget constraint. Use the gradient inequality for concave functions to note that $\log \tilde{c}_{it} \leq \log c_{it} + c_{it}^{-1}(\tilde{c}_{it} - c_{it})$. Then,

$$\begin{aligned} U(\tilde{c}_i) &= \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \log \tilde{c}_{it} \; dt\right] \\ &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \left(\log c_{it} + c_{it}^{-1}(\tilde{c}_{it} - c_{it}\right) dt\right] \\ &= U(c_i) + \mathbb{E}\left[\int_0^\infty \lambda \xi_{it}(\tilde{c}_{it} - c_{it}) dt\right] \leq U(c_i). \end{aligned}$$

The final inequality holds because $\mathbb{E}\left[\int_0^\infty \xi_{it}\tilde{c}_{it}\right] \leq n_{i0}$ since $\tilde{c}$ satisfies the intertemporal budget constraint in equation (38) while $\mathbb{E}\left[\int_0^\infty \xi_{it} c_{it}\right] = n_{i0}$ by construction.

We now want to derive the stochastic processes that these sufficient conditions imply. From the first-order condition (40), we obtain that $c_{it} > 0$ is an Itô process,

$$dc_{it}/c_{it} = \mu_{cit} dt + \sigma_{cit} dB_{yit}.$$

Using Itô's lemma we obtain

$$d(e^{-\rho_e t} c_{it}^{-1}) = d(\lambda \xi_{it}) \tag{41}$$

$$e^{-\rho_e t} c_{it}^{-1}\left(-\rho_e - \mu_{cit} + \sigma_{cit}^2\right) dt - e^{-\rho_e t} c_{it}^{-1} \sigma_{cit} dB_{yit} = -\lambda \xi_{it} r_t dt - \lambda \xi_{it} \frac{(z_{it} - w_t)^+}{\phi \sigma_y} dB_{yit}. \tag{42}$$

Matching terms we get the Euler equation and the first-order condition for labor,

$$\mu_{cit} = r_t - \rho_e + \sigma_{cit}^2, \tag{43}$$

$$\sigma_{cit} = \frac{(z_{it} - w_t)^+}{\phi \sigma_y}. \tag{44}$$

And any plan $c_i > 0$ satisfying these conditions and $c_{i0} = \lambda^{-1} = \rho_e n_{i0}$ satisfies FOC (40) and the intertemporal budget constraint with equality and is therefore optimal in the relaxed problem.

It only remains to show that our candidate plan c_i can indeed be implemented as part of a plan (c_i, ℓ_i) that satisfies the dynamic budget constraint. We set

$$\ell_{it} = \frac{\sigma_{cit} c_{it}}{\sigma_y \phi \rho_e} \geq 0, \tag{45}$$

We now define the process for net worth $n_{it} = c_{it}/\rho_e > 0$ which satisfies the initial condition by construction. Using the optimality conditions we compute

$$\begin{aligned} dn_{it}/n_{it} &= \left(r_t - \rho_e + \sigma_{cit}^2\right) dt + \sigma_{cit} dB_{yit}, \\ &= \left(r_t - c_{it}/n_{it} + \frac{\ell_{it}}{n_{it}} \sigma_y \phi \frac{(z_{it} - w_t)^+}{\phi \sigma_y}\right) dt + \frac{\ell_{it}}{n_{it}} \sigma_y \phi dB_{yit}, \\ &= \left(r_t - c_{it}/n_{it} + \frac{\ell_{it}}{n_{it}} (z_{it} - w_t)\right) dt + \frac{\ell_{it}}{n_{it}} \sigma_y \phi dB_{yit}, \end{aligned}$$

since $\ell_{it} = 0$ if $z_{it} < w_t$ by construction in equation (45).

Rearranging we obtain the dynamic budget constraint,

$$dn_{it} = (r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t))dt + \ell_{it} \sigma_y \phi dB_{yit}.$$

Lemma 1. *Assume non-negative adapted processes (c_i, ℓ_i) satisfy (43)-(45) with $c_{i0} = \rho_e n_{i0}$. Then the plan (c_i, ℓ_i) and the associated process $n_{it} = c_{it}/\rho_e$ satisfies the dynamic budget constraint subject to initial wealth and the natural borrowing limit and achieves the maximum utility of the relaxed intertemporal problem. It is therefore an optimal solution of the entrepreneur's problem.*

B Worker Optimality

The worker's problem is to choose a non-negative processes (c_w, ℓ) to maximize utility $U_w(c_w, \ell)$ subject to the dynamic budget constraint

$$dn_{wt} = (n_{wt} r_t - c_{wt} + w_t \ell_t) dt,$$

with initial wealth n_{w0} given and natural borrowing limit $n_{wt} \geq -\int_t^\infty e^{-\int_t^s r_u du} w_s \ell_s ds$.

This is a standard problem. First, for any plan (c_w, ℓ) satisfying the dynamic budget constraint and natural borrowing limit we satisfy the intertemporal budget constraint

$$\int_0^\infty e^{-\int_0^t r_u du}(c_{wt} - w_t \ell_t)dt \leq n_{w0}. \tag{46}$$

To see this compute

$$d(e^{-\int_0^t r_u du} n_{wt}) = e^{-\int_0^t r_u du} \left(-n_{wt} r_t + n_{wt} r_t - c_{wt} + w_t \ell_t\right) dt.$$

Integrating we obtain

$$\int_t^T e^{-\int_t^s r_u du}(c_{ws} - w_s \ell_s)ds + e^{-\int_t^T r_u du} n_{wT} = n_{w0}.$$

Use the natural borrowing limit to obtain

$$\int_t^T e^{-\int_t^s r_u du} c_{ws} ds - e^{-\int_t^\infty r_u du} w_s \ell_s ds \leq n_{w0},$$

and taking the limit $T \to \infty$ we obtain the intertemporal budget constraint. We can therefore consider the relaxed problem of maximizing utility subject to the intertemporal budget constraint.

The first-order conditions of the relaxed problem are

$$e^{-\rho_w t} c_{wt}^{-1} = \lambda e^{-\int_0^t r_u du} \tag{47}$$

$$e^{-\rho_w t} \ell_t^{1/\psi} = \lambda e^{-\int_0^t r_u du} w_t \iff \ell_t^{1/\psi} = c_{wt}^{-1} w_t, \tag{48}$$

and λ is set so the intertemporal budget constraint (46) holds with equality. These conditions are also sufficient for optimality.

From (47), taking time-derivatives, we obtain the Euler equation

$$dc_{wt}/c_{wt} = (r_t - \rho_w)dt. \tag{49}$$

A plan (c_w, ℓ) satisfying the first-order conditions (49) and (48) and the intertemporal budget constraint with equality, achieves the optimum of the relaxed problem.

It remains to show this plan can be implemented with the dynamic budget constraint. Set

$$n_{wt} = \int_t^{\infty} e^{-\int_t^s r_u du}(c_{ws} - w_s \ell_s)ds, \tag{50}$$

which satisfies the initial condition by construction. Taking derivatives with respect to time, we obtain the dynamic budget constraint

$$dn_{wt} = (n_{wt} r_t - c_{wt} + w_t \ell_t)dt.$$

Since $c_w \geq 0$, the natural borrowing limit is automatically satisfied.

Lemma 2. *Assume non-negative adapted processes* (c_w, ℓ) *satisfy* (49) *and* (48) *and the intertemporal budget constraint with equality.Then the plan* (c_w, ℓ) *and the associated process* n_w *given by* (50) *satisfies the dynamic budget constraint subject to initial wealth and the natural borrowing limit and achieves the maximum utility of the relaxed intertemporal problem. It is therefore an optimal solution of the worker's problem.*

C Proof of Proposition 1

We want to show that the prices and allocations $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ constructed as described in Section 2.3 constitute a competitive equilibrium. To do so we need to show that allocations and prices are in their permissible domain, that markets clear, and that agents are solving their problem taking prices as given.

C.1 Market Clearing

First, since $w_t, c_{it}, c_{wt} > 0$, we have that $\ell_{it} = \sigma_{ct}(z_{it})\frac{c_{it}}{\rho_e \phi \sigma_y} \geq 0$ and $\ell_t = c_{wt}^{-1} w_t > 0$. Furthermore, by construction $\int_0^1 n_{it} di + n_{wt} = 0$.

To show that markets clear we can use that the KFE is derived from the Euler equations of entrepreneurs and the worker, yielding that

$$\frac{\int_{\{z_{it}=z\}} c_{it} di}{c_{wt}} = \frac{\eta_{zt}(z)}{\eta_{wt}}.$$

Integrating across z and using $\eta_{et} = 1 - \eta_{wt}$ and $c_{wt} = (1 - \eta_{et}) Z_t \ell_t$, we obtain

$$\int_0^1 c_{it} di = \eta_{et} \frac{c_{wt}}{1 - \eta_{et}} = \eta_{et} Z_t \ell_t,$$

and therefore

$$\int_0^1 c_{it} di + c_{wt} = Z_t \ell_t.$$

Equations (18) and (19) then ensure that market clearing (equations 3 and 4) hold.

C.2 Optimality

It remains to show that the allocations (c_i, ℓ_i) and (c_w, ℓ) satisfy the dynamic budget constraints, natural borrowing constraints, and are optimal given prices (r, w).

Since we constructed the entrepreneurs' allocations (c_i, ℓ_i) to satisfy the first-order conditions and $c_{i0} = \rho_e n_{i0}$, Lemma 1 in Appendix A ensures they satisfy budget constraints and are optimal.

The allocation satisfies the first order conditions for the worker. So given Lemma 2 in Appendix B, we only need to show that it satisfies the worker's intertemporal budget constraint with equality. We defined $n_{wt} = -\int_0^1 n_{it} di$. We therefore have that

$$
\begin{aligned}
e^{-\int_t^T r_u du} n_{wT} &= -\int_0^1 e^{-\int_t^T r_u du} n_{it} di, \\
&= \int_0^1 \left(-n_{it} + \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds - \int_t^T e^{-\int_t^s r_u du} \ell_{is} \phi \sigma_y dB_{iys} \right) di, \\
&= \int_0^1 -n_{it} di + \int_0^1 \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds di - \int_0^1 \int_t^T e^{-\int_t^s r_u du} \ell_{is} \phi \sigma_y dB_{iys} di \\
&= \left(n_{wt} + \int_0^1 \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds di \right).
\end{aligned}
$$

where the last term is eliminated imposing a law of large numbers.[5] We can switch the order of integration using Tonelli to obtain,

$$
e^{-\int_t^T r_u du} n_{wT} = n_{wt} + \int_t^T e^{-\int_t^s r_u du} \left(\int_0^1 c_{is} di - \int_0^1 z_{is} \ell_{is} + w_s \int_0^1 \ell_{is} di \right) ds.
$$

Use the resource constraints,

$$
e^{-\int_t^T r_u du} n_{wT} = n_{wt} + \int_t^T e^{-\int_t^s r_u du} (-c_{ws} + w_s \ell_s) ds.
$$

Take the limit $T \to \infty$ and use that $\lim_{T \to \infty} e^{-\int_0^T r_u du} n_{wT} = 0$ to obtain

$$
\int_t^\infty e^{-\int_t^s r_u du} (c_{ws} - w_s \ell_s) ds = n_{wt}. \tag{51}
$$

As desired.

C.3 Conclusion

Thus, initial wealth n_{w0} and n_{i0} and allocations and prices $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ that satisfy the proposition assumptions and are constructed by the detailed procedure are a *Competitive Equilibrium.*

[5] The assumption that a LLN holds is the same one that we impose when writing the goods resource constraint in equation (3).

D The Social Planner's Transformed Objective Function

We start with the planner's objective function

$$\mathbb{E}\left[\Gamma_w \int_0^\infty e^{-\rho_w t}\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right)dt + \Gamma_e \int_0^1 \left(\int_0^\infty e^{-\rho_e t}\log c_{it}dt\right)di\right].$$

First, take the utility of an entrepreneur,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_e t}\log c_{it}dt\right].$$

Use the Euler equation to write

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_e t}\left[\log c_{i0} + \log\left(e^{\int_0^t (r_s-\rho_e+\sigma_{cis}^2-\frac{1}{2}\sigma_{cis}^2)ds+\int_0^t \sigma_{cis}dB_{is}}\right)\right]dt\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}e^{-(\rho_e-\rho_w)t}\left[\log c_{i0} + \log\left(e^{\int_0^t (r_s-\rho_w+(\rho_w-\rho_e)+\sigma_{cis}^2-\frac{1}{2}\sigma_{cis}^2)ds+\int_0^t \sigma_{cis}dB_{is}}\right)\right]dt\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}e^{-(\rho_e-\rho_w)t}\left[\log c_{i0} + \log\left(e^{\int_0^t (r_s-\rho_w)ds}e^{\int_0^t ((\rho_w-\rho_e)+\sigma_{cis}^2-\frac{1}{2}\sigma_{cis}^2)ds+\int_0^t \sigma_{cis}dB_{is}}\right)\right]dt\right].$$

Use the worker's Euler equation to replace $e^{\int_0^t (r_s-\rho_w)ds} = c_{wt}/c_{w0}$,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}e^{-(\rho_e-\rho_w)t}\left[\log c_{i0} + \log\left(\frac{c_{wt}}{c_{w0}}e^{\int_0^t ((\rho_w-\rho_e)+\sigma_{cis}^2-\frac{1}{2}\sigma_{cis}^2)ds+\int_0^t \sigma_{cis}dB_{is}}\right)\right]dt\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}e^{-(\rho_e-\rho_w)t}\left[\log c_{i0} - \log c_{w0} + \log c_{wt} + \left(\int_0^t ((\rho_w-\rho_e)+\sigma_{cis}^2 - \frac{1}{2}\sigma_{cis}^2)ds + \int_0^t \sigma_{cis}dB_{is}\right)\right]dt\right].$$

Take the expectation of the stochastic integral $\mathbb{E}\left[\int_0^t \sigma_{cis}dB_{is}\right] = 0$ and simplify the terms involving σ_{cis}^2,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + \int_0^t ((\rho_w-\rho_e)+\frac{1}{2}\sigma_{cis}^2)ds\right)dt\right].$$

For the next steps, it's cleaner if we first split the expression in two and then bring it back together.

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t} e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + (\rho_w - \rho_e)t\right) dt + \int_0^\infty e^{-\rho_e t}\int_0^t \frac{1}{2}\sigma^2_{cis} ds dt\right].$$

Now flip the integration for the second term,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t} e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + (\rho_w - \rho_e)t\right) dt + \int_0^\infty e^{-\rho_e t}\int_t^\infty e^{-\rho_e(s-t)}\frac{1}{2}\sigma^2_{cit} ds dt\right].$$

The advantage is that the integrand in the inside integral has a constant σ^2_{cit}. We can solve out the integrals in the second term,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t} e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + (\rho_w - \rho_e)t\right) dt + \int_0^\infty e^{-\rho_e t}\frac{1}{\rho_e}\frac{1}{2}\sigma^2_{cit} dt\right].$$

Put it all back together,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t} e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + (\rho_w - \rho_e)t + \frac{1}{\rho_e}\frac{1}{2}\sigma^2_{cit}\right) dt\right].$$

Now plug this back into the objective function of the planner,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}\Gamma_w\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt + \right.$$
$$\left.\int_0^1\int_0^\infty \Gamma_e e^{-\rho_w t} e^{-(\rho_e-\rho_w)t}\left(\log c_{i0} - \log c_{w0} + \log c_{wt} + (\rho_w - \rho_e)t + \frac{1}{\rho_e}\frac{1}{2}\sigma^2_{cit}\right) dt di\right].$$

Group terms

$$\mathbb{E}\left[\int_0^\infty e^{-\rho_w t}\left(\log c_{wt}\left(\Gamma_w + \Gamma_e e^{-(\rho_e-\rho_w)t}\right) - \Gamma_w\frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt\right]$$
$$+\mathbb{E}\left[\int_0^1\int_0^\infty \Gamma_e e^{-\rho_e t}\frac{1}{\rho_e}\frac{1}{2}\sigma^2_{cit} dt di\right] + \int_0^1 \Gamma_e\int_0^\infty e^{-\rho_e t}(\log c_{i0} - \log c_{w0} + (\rho_w - \rho_e)t) dt di.$$

Drop the expectation in the first term and pull out the Γ_w, use a law of large numbers to drop the expectation in the second term and replace $\sigma_{cit} = \sigma_c(z_{it}; \eta_{zt})$

$$\int_0^\infty e^{-\rho_w t}\Gamma_w \left(\log c_{wt} \left(\frac{\Gamma_w + \Gamma_e e^{-(\rho_e - \rho_w)t}}{\Gamma_w} \right) - \frac{\ell_t^{1+1/\psi}}{1+1/\psi} \right) dt$$
$$+ \int_0^1 \int_0^\infty \Gamma_e e^{-\rho_e t} \frac{1}{\rho_e} \frac{1}{2} \sigma_c(z_{it}; \eta_{zt})^2 dt di + \int_0^1 \Gamma_e \int_0^\infty e^{-\rho_e t} (\log c_{i0} - \log c_{w0} + (\rho_w - \rho_e)t) dt di.$$

The last term $\int_0^1 \Gamma_e \int_0^\infty e^{-\rho_e t}(\rho_w - \rho_e) t dt di = \Gamma_e \frac{\rho_w - \rho_e}{\rho_e^2}$ is a constant, so we can drop it for optimization purposes.

Use $c_{i0} = \eta_{i0} c_0$ and $c_{w0} = \eta_{w0} c_0$ and compute the integral in the third term,

$$\int_0^\infty e^{-\rho_w t}\Gamma_w \left(\log c_{wt} \left(\frac{\Gamma_w + \Gamma_e e^{-(\rho_e - \rho_w)t}}{\Gamma_w} \right) - \frac{\ell_t^{1+1/\psi}}{1+1/\psi} \right) dt \tag{52}$$
$$+ \int_0^1 \int_0^\infty \Gamma_e e^{-\rho_e t} \frac{1}{\rho_e} \frac{1}{2} \sigma_c(z_{it}; \eta_{zt})^2 dt di + \Gamma_e \frac{\int_0^1 \log \eta_{i0} di - \log \eta_{w0}}{\rho_e}.$$

This gives us the objective function (24).

The KFE comes from the Euler equations and the resource constraint, and $c_{wt} = (1 - \eta_{et}) Z(\eta_{zt}) \ell_t$ from the resource constraint.

E Proof of Proposition 2

The allocation maximizes the social planner's transformed objective (24). To show it is an optimal allocation we only need to show that it is implementable. An argument analogous to the one in Proposition 1 shows that $(c_w, \ell, c_i, \ell_i, r, w, \tau, T, n_w, n_i)$ is a competitive equilibrium with tax and lump-sum transfers, so $\mathcal{S}$ is implementable. Since $w_t(1 - \tau_t), c_{it}, c_{wt} > 0$, we have $\ell_{it} \geq 0$ and $\ell_t > 0$, and the first order conditions for the workers and entrepreneurs are satisfied and $n_{wt} + \int_0^1 n_{it} di = 0$ by construction. The same argument as in the proof of Proposition 1 shows that the resource constraints hold and that the dynamic budget constraint of entrepreneurs hold with the natural borrowing limit and the intertemporal budget constraints hold with equality, which imply that the plan (c_i, ℓ_i) is optimal for entrepreneur i. Since $T_t = \tau_t w_t \ell_t$, the budget constraint of the worker is the same as in the case without taxes, so the same argument as in the proof of Proposition 1 shows that the workers dynamic budget constraint and natural borrowing limit hold, and the intertemporal budget constraint holds with equality, which implies the plan (c_w, ℓ) is optimal for the worker. We conclude that $(c_w, \ell, c_i, \ell_i, r, w, \tau, T, n_w, n_i)$ is a competitive equilibrium with tax and lump-sum transfers, so $\mathcal{S}$ is implementable.

F Derivation of KFE for $\eta_z(z,t)$

F.1 Derive process for η_{it}

Keep track of (z_i, η_i) where $\eta_{it} = c_{it}/c_t$. We need to compute the growth rate of aggregate consumption c_t:

$$c_{wt}\mu_{cwt} + \int c_{it}\mu_{cit}di = \mu_{ct}c_t$$

$$\eta_{wt}\mu_{ct} + \int \eta_{zt}\mu_{ct}(z)dz = \mu_{ct}$$

(using the fact that growth rates depend only on z type).

Now use the Euler equations to re write this as

$$\eta_{wt}(r_t - \rho) + \int \eta_{zt}\left(r_t - \rho_e + \left(r_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}\right)\right)dz = \mu_c$$

$$\Bigg(\underbrace{\eta_{wt} + \int \eta_{zt}dz}_{=1}\Bigg) r_t - \underbrace{\left(\eta_{wt}\rho + \rho_e \int \eta_{zt}dz\right)}_{=\bar{\rho}_t} + \int \eta_{zt}\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz = \mu_c$$

$$\mu_{ct} = r_t - \bar{\rho}_t + \int \eta_{zt}\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz.$$

So now we can compute the law of motion of $\eta_{it} = c_{it}/c_t$:

$$
\begin{aligned}
d(c_{it}/c_t) &= \left(\frac{c_{it}}{c_t}\right)(\mu_{cit} - \mu_{ct})\,dt + \left(\frac{c_{it}}{c_t}\right)\sigma_{cit} dB_{it} \\
d(c_{it}/c_t) &= \left(\frac{c_{it}}{c_t}\right)\underbrace{\left(\bar{\rho}_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2} - \int \eta_{zt}\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz\right)}_{\mu_\eta(z,t)} dt \\
&\quad + \left(\frac{c_{it}}{c_t}\right)\underbrace{\frac{z_t - w_t}{\phi\sigma_y}}_{\sigma_{ct}(z)} dB_{yt},
\end{aligned}
$$

or, more compactly,

$$
d\eta_{it} = \eta_{it}\mu_\eta(z,t)dt + \eta_{it}\sigma_{ct}(z)dB_{yt}.
$$

F.2 Derive KFE

Let $D(\eta, z, t)$ be the joint distribution of η, z. $D(\eta, z, t)$ is the measure of entrepreneurs with $z_i = z$ and $\eta_i = \eta$. This distribution has a convenient mapping into $\eta(z,t)$ as

$$
\eta(z,t) = \int D(\eta, z, t)\eta d\eta
$$

Let $X = (z_i, \eta_i)$

$$
dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_z(z) & 0 \\ 0 & \sigma_c(z,t)\eta \end{bmatrix}\begin{bmatrix} dB_{zt} \\ dB_{yt} \end{bmatrix}.
$$

In order to derive a KFE for the joint distribution $D(\cdot,\cdot)$, we re-write dX_t as a function of a standard multivariate Wiener process with independent component. Consider two independent Wiener processes W_{1t}, W_{2t}:

$$
\begin{aligned}
B_{zt} &= W_{1t} \\
B_{yt} &= \lambda_{yz}W_{1t} + \sqrt{1 - \lambda_{yz}^2}W_{2t}.
\end{aligned}
$$

Therefore, we have

$$dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_z(z) & 0 \\ \sigma_c(z,t)\eta\lambda_{yz} & \sigma_c(z,t)\eta\sqrt{1-\lambda_{yz}^2} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

To simplify the notation, we can write more generally

$$dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

We derive the KFE for $D(\eta, z, t)$, starting from the first order terms[6]

$$\begin{aligned}\partial D(\eta,z,t) =& -\frac{\partial}{\partial z}\left[\mu_z(z)D(\eta,z,t)\right] - \frac{\partial}{\partial \eta}\left[\mu_\eta(z,t)\eta D(\eta,z,t)\right] + \text{ 2nd order terms} \\ =& -\left[\mu_z'(z)D(\eta,z,t) + \mu_z(z)D_z'(\eta,z,t)\right] \\ & -\left[\mu_\eta(z)D(\eta,z,t) + \mu_\eta(z)\eta D_\eta'(\eta,z,t)\right] + \text{ 2nd order terms.}\end{aligned}$$

Now, we define the second order terms. The variance covariance matrix is

$$\Lambda = \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix} \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix}'$$

$$\Lambda = \begin{bmatrix} \sigma_{z1}^2(z) + \sigma_{z2}^2(z) & \eta\sigma_{z1}(z)\sigma_{c1}(z) + \eta\sigma_{z2}(z)\sigma_{c2}(z) \\ \eta\sigma_{z1}(z)\sigma_{c1}(z) + \eta\sigma_{z2}(z)\sigma_{c2}(z) & \sigma_{c1}^2(z)\eta^2 + \sigma_{c2}^2(z)\eta^2 \end{bmatrix},$$

$$\text{2nd order terms} = \frac{1}{2}\frac{\partial^2}{\partial z^2}[\Lambda_{11}D(\eta,z,t)] + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}[\Lambda_{22}D(\eta,z,t)] + \frac{\partial^2}{\partial \eta z}[\Lambda_{12}D(\eta,z,t)].$$

[6]Since $d\eta_{it}/\eta_{it} = dc_{it}/c_{it}$, we have $d\eta_i = \mu_c\eta_i dt + \sigma_c\eta_i dB_t$.

Further evaluating the matrix by elements, the first term is:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial z^2}[\Lambda_{11}D(\eta,z,t)] &= \frac{1}{2}\frac{\partial^2}{\partial z^2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))D(\eta,z,t)] \\
&= \frac{1}{2}\frac{\partial}{\partial z}[2(\sigma_{z1}(z)\sigma_{z1}'(z)+\sigma_{z2}(z)\sigma_{z2}'(z))D(\eta,z,t)+(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))D_z(\eta,z,t)] \\
&= \frac{\partial}{\partial z}[(\sigma_{z1}(z)\sigma_{z1}'(z)+\sigma_{z2}(z)\sigma_{z2}'(z))D(\eta,z,t)] \\
&\quad + \frac{1}{2}\frac{\partial}{\partial z}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))D_z(\eta,z,t)] \\
&= [\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)D(\eta,z,t)]+[\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)D_z(\eta,z,t)] \\
&\quad + [\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)D_z(\eta,z,t)]+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))D_{zz}(\eta,z,t)] \\
&= [\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)D(\eta,z,t)]+2[\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)D_z(\eta,z,t)] \\
&\quad + \frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))D_{zz}(\eta,z,t)].
\end{aligned}
$$

The second term is:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial \eta^2}[\Lambda_{22}D(\eta,z,t)] =& \frac{1}{2}\frac{\partial^2}{\partial \eta^2}[(\sigma_{c1}^2(z)\eta^2+\sigma_{c2}^2(z)\eta^2)D(\eta,z,t)] \\
=&(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\frac{1}{2}\frac{\partial^2}{\partial \eta^2}[\eta^2 D(\eta,z,t)] \\
=&(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\frac{1}{2}\frac{\partial}{\partial \eta}[2\eta D(\eta,z,t)+\eta^2 D_\eta(\eta,z,t)] \\
=&(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[\frac{\partial}{\partial \eta}[\eta D(\eta,z,t)]+\frac{1}{2}\frac{\partial}{\partial \eta}[\eta^2 D_\eta(\eta,z,t)]\right] \\
=&(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[D(\eta,z,t)+\eta D_\eta(\eta,z,t)]+\eta D_\eta(\eta,z,t)+\frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right] \\
=&(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[D(\eta,z,t)+2\eta D_\eta(\eta,z,t)]+\frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right].
\end{aligned}
$$

The third term is:

$$
\begin{aligned}
\frac{\partial^2}{\partial\eta\partial z}[\Lambda_{12} D(\eta,z,t)] &= \frac{\partial^2}{\partial\eta\partial z}[[\eta\sigma_{z1}(z)\sigma_{c1}(z)+\eta\sigma_{z2}(z)\sigma_{c2}(z)]D(\eta,z,t)] \\
&= \frac{\partial^2}{\partial\eta\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))\eta D(\eta,z,t)] \\
&= \frac{\partial}{\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))D(\eta,z,t)] + \frac{\partial}{\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))\eta D_\eta(\eta,z,t)] \\
&= \sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]D(\eta,z,t) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z)D_z(\eta,z,t)\right] \\
&+ \sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]\eta D_\eta(\eta,z,t) \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))\eta D_{\eta z}(\eta,z,t)\right].
\end{aligned}
$$

Combining all first and second order terms yields

$$
\begin{aligned}
\partial D(\eta,z,t) =& -\left[\mu'_z(z)D(\eta,z,t)+\mu_z(z)D'_z(\eta,z,t)\right] - \left[\mu_\eta(z)D(\eta,z,t)+\mu_\eta(z)\eta D'_\eta(\eta,z,t)\right] \\
&+ \left(\sum_j \sigma'_{zj}(z)^2+\sigma_{zj}(z)\sigma''_{zj}(z)\right)D(\eta,z,t) + 2\left(\sum_j \sigma_{zj}(z)\sigma'_{zj}(z)\right)D_z(\eta,z,t) \\
&+ \frac{1}{2}[(\sigma^2_{z1}(z)+\sigma^2_{z2}(z))D_{zz}(\eta,z,t)] \\
&+(\sigma^2_{c1}(z)+\sigma^2_{c2}(z))\left[[D(\eta,z,t)+2\eta D_\eta(\eta,z,t)]+\frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right] \\
&+ \sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]D(\eta,z,t) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z)D_z(\eta,z,t)\right] \\
&+ \sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]\eta D_\eta(\eta,z,t) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))\eta D_{\eta z}(\eta,z,t)\right].
\end{aligned}
$$

To obtain $\eta(z,t)$, use that

$$\begin{aligned}
\eta(z,t) &= \int D(\eta,z,t)\eta d\eta \\
\partial\eta(z,t) &= \int \partial D(\eta,z,t)\eta d\eta \\
\eta_z(z,t) &= \int D_z(\eta,z,t)\eta d\eta.
\end{aligned}$$

So,

$$\begin{aligned}
\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right] - \left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)\int \eta\eta D_\eta'(\eta,z,t)d\eta\right] \\
&+\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t) \\
&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)] \\
&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2\int \eta\eta D_\eta(\eta,z,t)d\eta]+\frac{1}{2}[\int \eta\eta^2 D_{\eta\eta}(\eta,z,t)d\eta]\right] \\
&+\sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)] \\
&+\sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\int \eta\eta D_\eta(\eta,z,t)d\eta+\sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))\int \eta\eta D_{\eta z}(\eta,z,t)d\eta\right].
\end{aligned}$$

Rearranging yields

$$
\begin{aligned}
\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right]-\left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)\int\eta^2 D_\eta'(\eta,z,t)d\eta\right]\\
&+\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t)\\
&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)]\\
&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2\int\eta^2 D_\eta(\eta,z,t)d\eta]+\frac{1}{2}[\int\eta^3 D_{\eta\eta}(\eta,z,t)d\eta]\right]\\
&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right]\\
&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\int\eta^2 D_\eta(\eta,z,t)d\eta+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z))\int\eta^2 D_{\eta z}(\eta,z,t)d\eta\right].
\end{aligned}
$$

Substitute

$$
\int\eta^2 D_\eta(z,\eta,t)d\eta=\left.\eta^2 D(z,\eta,t)\right|_0^1-2\int_0^1\eta D(z,\eta,t)d\eta=-2\eta(z,t)
$$

and conjecture that $\left.\eta^2 D(z,\eta,t)\right|_0^1=\lim_{\eta\to1}\eta^2 D(z,\eta,t)-\lim_{\eta\to0}\eta^2 D(z,\eta,t)=0-0=0$ in the KFE to obtain

$$
\begin{aligned}
\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right]-\left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)(-2\eta(z,t))\right]\\
&+\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t)\\
&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)]\\
&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2(-2\eta(z,t))]+\frac{1}{2}[\int\eta^3 D_{\eta\eta}(\eta,z,t)d\eta]\right]\\
&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right]\\
&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z,t))+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z))\int\eta^2 D_{\eta z}(\eta,z,t)d\eta\right].
\end{aligned}
$$

Finally, use integration by part to obtain

$$\int \eta^3 D_{\eta\eta}(\eta, z, t)d\eta = \eta^3 D_\eta(z, \eta, t)\big|_0^1 - 3\int \eta^2 D_\eta(\eta, z, t)d\eta = 6\eta(z, t)$$

and conjecture that $\eta^3 D_\eta(z, \eta, t)\big|_0^1 = \lim_{\eta\to 1}\eta^3 D_\eta(z, \eta, t) - \lim_{\eta\to 0}\eta^3 D_\eta(z, \eta, t) = 0 - 0 = 0$. Again, substitute these expressions in the KFE to obtain

$$\begin{aligned}
\partial\eta(z, t) = &- [\mu_z'(z)\eta(z, t) + \mu_z(z)\eta_z'(\eta, z, t)] - [\mu_\eta(z)\eta(z, t) + \mu_\eta(z)(-2\eta(z, t))] \\
&+ \left(\sum_j \sigma_{zj}'(z)^2 + \sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z, t) + 2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z, t) \\
&+ \frac{1}{2}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))\eta_{zz}''(\eta, z, t)] \\
&+ (\sigma_{c1}^2(z) + \sigma_{c2}^2(z))\left[[\eta(z, t) + 2(-2\eta(z, t))] + \frac{1}{2}[6\eta(z, t)]\right] \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z, t) + \sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z, t)] \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z, t)) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))\int \eta^2 D_{\eta z}(\eta, z, t)d\eta\right].
\end{aligned}$$

A similar argument can be applied to obtain

$$\int \eta^2 D_{\eta z}(\eta, z, t)d\eta = -2\eta_z'(z, t),$$

which implies that

$$
\begin{aligned}
\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right]-\left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)(-2\eta(z,t))\right]\\
&+\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t)\\
&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)]\\
&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2(-2\eta(z,t))]+\frac{1}{2}[6\eta(z,t)]\right]\\
&+\sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right]\\
&+\sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z,t))+\sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))(-2\eta_z'(z,t))\right].
\end{aligned}
$$

Grouping terms yields

$$
\begin{aligned}
\partial\eta(z,t) = \eta(z,t)&\left[-\mu_z'(z)+\mu_\eta(z)+\left(\sum_j \sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)-\sum_j\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\right]\\
&+\eta_z'(z,t)\left[-\mu_z(z)+2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)-\sum_j \sigma_{zj}(z)\sigma_{cj}(z)\right]\\
&+\eta_{zz}''(\eta,z,t)\left[\frac{1}{2}(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\right].
\end{aligned}
$$

In our model

$$
\begin{aligned}
\sigma_{z1}(z) &= \sigma_z(z)\\
\sigma_{z2}(z) &= 0\\
\sigma_{c1}(z) &= \sigma_c(z,t)\lambda_{yz}\\
\sigma_{c2}(z) &= \sigma_c(z,t)\sqrt{1-\lambda_{yz}^2}\\
\sigma_c(z,t) &= \frac{z_t-w_t}{\phi\sigma_y}.
\end{aligned}
$$

Therefore, the KFE is

$$
\begin{aligned}
\partial\eta(z,t) = \eta(z,t) & \left[-\mu_z'(z) + \mu_\eta(z) + \sigma_z'(z)^2 + \sigma_z(z)\sigma_z''(z) - \right. \\
& \left. [\sigma_z'(z)\sigma_c(z) + \sigma_z(z)\sigma_c'(z)] \left(\lambda_{yz} + \sqrt{1-\lambda_{yz}^2}\right)\right] \\
& + \eta_z'(z,t)\left[-\mu_z(z) + 2\sigma_z(z)\sigma_z'(z) - \sigma_z(z)\sigma_c(z)\left(\lambda_{yz} + \sqrt{1-\lambda_{yz}^2}\right)\right] \\
& + \eta_{zz}''(\eta,z,t)\left[\frac{1}{2}\sigma_z^2(z)\right]
\end{aligned}
$$

where $\mu_z(z), \sigma_z(s), \lambda_{yz}$ are primitives, and $\sigma_c(z), \mu_\eta(z)$ are equilibrium objects given by:

$$
\begin{aligned}
\sigma_c(z,t) &= \frac{z_t - w_t}{\phi\sigma_y} \\
\mu_\eta(z,t) &= \bar{\rho}_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2} - \int \eta_{zt}\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz.
\end{aligned}
$$

G Numerical Example Details

G.1 Process for z

Assume $\log z$ follows

$$
d\log z_{it} = \theta\left(\log\bar{z} - \log z_{it}\right)dt + \sigma_{\log z}dB_{zt} +
$$

where $\sigma_{\log z}$ is a constant. Also set $\log(\bar{z}) = -\frac{1}{\theta}\frac{1}{2}\sigma_{\log z}^2$. Then, z follows

$$
\begin{aligned}
dz_{it} &= z_{it}\left(-\theta\log\left(z_{it}\right) + \theta\log(\bar{z}) + \frac{1}{2}\sigma_{\log z}^2\right)dt + z_{it}\sigma_{\log z}dB_{zt} \\
&= -\theta z_{it}\log\left(z_{it}\right)dt + z_{it}\sigma_{\log z}dB_{zt}
\end{aligned}
$$

so that we have

$$\mu_z(z) = -\theta z \log(z)$$

$$\mu_z'(z) = -\theta(\log(z) + 1)$$

$$\sigma_z(z) = z\sigma_{\log z}$$

$$\sigma_z'(z) = \sigma_{\log z}$$

$$\sigma_z''(z) = 0$$

G.2 Calibration

Parameter	Description	Value
ψ	Frisch Elasticity	1
ρ	Household's discount factor	0.04
ρ_e	Entrepreneurs' discount factor	0.2
θ	Speed of mean reversion of z_{it}	0.3
σ_y	Volatility of innovations to output, B_{yt}	0.15
$\sigma_{\log z}$	Volatility of innovations to $\log(z)$, B_{zt}	0.03
λ_{yz}	Correlation of B_{yt}, B_{zt}	0.2
ϕ	Exposure to idiosyncratic risk	1