Appendix (For Online Publication)

A Proposition 1 (the Optimal Allocation)

A.1 Defining The Current Value Hamiltonian

The social planner problem is :

$$\begin{aligned}
&\max_{\{L_{pt}, x_{it}, \tilde{x}_{it}\}} \int_0^\infty e^{-(\rho - g_L)} L_0 u(c_t, x_{it}, \tilde{x}_{it}) dt \\
s.t. c_t &= \frac{Y_t}{L_t} \\
Y_t &= N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}} \right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \\
\dot{N}_t &= \frac{1}{\chi}(L_t - L_{pt}) \\
L_t &= L_0 e^{g_L t}
\end{aligned}$$

First, replace c_t in the problem:

$$\begin{aligned}
&\max_{\{L_{pt}, x_{it}, \tilde{x}_{it}\}} \int_0^\infty e^{-(\rho - g_L)} L_0 u\left(\frac{Y_t}{L_t}, x_{it}, \tilde{x}_{it}\right) dt \\
s.t. Y_t &= N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}} \right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \\
\dot{N}_t &= \frac{1}{\chi}(L_t - L_{pt}) \\
L_t &= L_0 e^{g_L t}
\end{aligned}$$

Next, define Hamiltonian with state variable N_t, control variables $\{L_{pt}, x_{it}, \tilde{x}_{it}\}$ and co-state variable μ:

$$H_t(L_{pt}, x_{it}, \tilde{x}_{it}, N_t, \mu_t) = u\left(\frac{Y_t}{L_t}, x_{it}, \tilde{x}_{it}\right) + \mu_t \dot{N}_t$$

Or also:

$$H_t(L_{pt}, x_{it}, \tilde{x}_{it}, N_t, \mu_t) = u\left(\frac{Y_t}{L_t}, x_{it}, \tilde{x}_{it}\right) + \mu_t \frac{1}{\chi}(L_t - L_{pt})$$

A.2 The First-Order Conditions of the Hamiltonian

The FOC are:

$$\begin{cases} \frac{\partial H}{\partial L_{pt}} = 0 \\ \frac{\partial H}{\partial x_{it}} = 0 \\ \frac{\partial H}{\partial \tilde{x}_{it}} = 0 \\ \frac{\partial H}{\partial N_t} = (\rho - g_L)\mu_t - \dot{\mu}_t \end{cases}$$

Start with $\frac{\partial H}{\partial L_{pt}} = 0$ and recall the definition of the utility function, $u(c_t, x_{it}, \tilde{x}_{it}) = log(c_t) - \frac{\kappa}{2}\frac{1}{N_t^2}\int_0^{N_t} x_{it}^2 di - \frac{\tilde{k}}{2}\frac{1}{N_t}\int_0^{N_t} \tilde{x}_{it}^2 di$. Then:

$$\begin{aligned}
\frac{\partial H}{\partial L_{pt}} &= \left(\frac{\partial u}{\partial(\frac{Y_t}{L_t})}\right)\left(\frac{\partial\left(\frac{Y_t}{L_t}\right)}{\partial L_{pt}}\right) - \frac{\mu_t}{\chi} \\
&= \left(\frac{1}{\left(\frac{Y_t}{L_t}\right)}\right)\left(\frac{N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}}}{L_t}\frac{1}{1-\eta}L_{pt}^{\frac{\eta}{1-\eta}}\right) - \frac{\mu_t}{\chi} \\
&= \left[N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}\right]^{-1} N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}}\frac{1}{1-\eta}L_{pt}^{\frac{\eta}{1-\eta}} - \frac{\mu_t}{\chi} \\
&= L_{pt}^{-\frac{1}{1-\eta}}\frac{1}{1-\eta}L_{pt}^{\frac{\eta}{1-\eta}} - \frac{\mu_t}{\chi} \\
&= \frac{L_{pt}^{-1}}{1-\eta} - \frac{\mu_t}{\chi}
\end{aligned}$$

Thus the FOC is:

$$\frac{L_{pt}^{-1}}{1-\eta} = \frac{\mu_t}{\chi}$$

and therefore:

$$g_\mu = -g_{L_p}$$

Next, we compute $\frac{\partial H}{\partial x_{it}} = 0$. The derivative of the Hamiltonian is given by:

$$\begin{aligned}
\frac{\partial H}{\partial x_{it}} &= \frac{\partial}{\partial x_{it}} u\left(\frac{Y_t}{L_t}, x_{it}, \tilde{x}_{it}\right) \\
&= \frac{\partial}{\partial x_{it}} u\Big(\frac{N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}, x_{it}, \tilde{x}_{it}\Big) \\
&= \frac{1}{\frac{N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t}+(1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}} \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}}}{L_t} \frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}-1} \frac{\alpha}{N_t} - \frac{\partial}{\partial x_{it}}\left(\frac{\kappa}{2N_t^2}\int_0^{N_t} x_{it}^2 di\right) \\
&= \frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} \frac{\alpha}{N_t} - \frac{\partial}{\partial x_{it}}\left(\frac{\kappa}{2N_t^2}\int_0^{N_t} x_{it}^2 di\right)
\end{aligned}$$

Using symmetry, $\int_0^{N_t} x_{it}^2 di = x_{it}^2 N_t$. Hence, the FOC $\frac{\partial H}{\partial x_{it}} = 0$ is:

$$\frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} \alpha = \kappa x_{it}$$

Next, we compute $\frac{\partial H}{\partial \tilde{x}_{it}} = 0$. The derivative of the Hamiltonian is given by:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{x}_{it}} &= \frac{\partial}{\partial \tilde{x}_{it}} u\left(\frac{Y_t}{L_t}, x_{it}, \tilde{x}_{it}\right) \\
&= \frac{\partial}{\partial \tilde{x}_{it}} u\Big(\frac{N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}, x_{it}, \tilde{x}_{it}\Big) \\
&= \frac{1}{\frac{N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_{it}}{N_t}+(1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}} \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}}}{L_t} \frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}-1} (1-\alpha) - \tilde{\kappa}\tilde{x}_{it} \\
&= \frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} (1-\alpha) - \tilde{\kappa}\tilde{x}_{it}
\end{aligned}$$

Therefore the FOC $\frac{\partial H}{\partial \tilde{x}_{it}} = 0$ is:

$$\frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} (1-\alpha) = \tilde{\kappa}\tilde{x}_{it}$$

Finally, the last FOC is $\frac{\partial H}{\partial N_t} = (\rho - g_L)\mu_t - \dot{\mu}_t$. The derivative of the Hamiltonian $\frac{\partial H}{\partial N_t}$ is:

$$\frac{\partial H}{\partial N_t} = \frac{\partial}{\partial N_t} u(\frac{N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}, x_{it}, \tilde{x}_{it})$$

$$= \frac{1}{\frac{N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}}{L_t}} \frac{L_{pt}^{\frac{1}{1-\eta}}}{L_t} \left[\frac{1}{\sigma-1} N_t^{\frac{1}{\sigma-1}-1} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} \dots \dots\right.$$

$$\left.\dots + \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}-1} \left(-\frac{\alpha x_{it}}{N_t^2}\right) N_t^{\frac{1}{\sigma-1}}\right] - \frac{\partial}{\partial N_t}\left(\frac{\kappa}{2}\frac{1}{N_t^2}\int_0^{N_t} x_{it}^2 di\right) - -\frac{\partial}{\partial N_t}\left(\frac{\tilde{k}}{2}\frac{1}{N_t}\int_0^{N_t} \tilde{x}_{it}^2 di\right)$$

$$= N_t^{-1}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}\left[\frac{1}{\sigma-1}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right) - \frac{\alpha x_{it}}{N_t}\right] \dots$$

$$\dots - \frac{\partial}{\partial N_t}\left(\frac{\kappa}{2}\frac{1}{N_t^2}\int_0^{N_t} x_{it}^2 di\right) - -\frac{\partial}{\partial N_t}\left(\frac{\tilde{k}}{2}\frac{1}{N_t}\int_0^{N_t} \tilde{x}_{it}^2 di\right)$$

$$= \left[\frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}\right] - \frac{\partial}{\partial N_t}\left(\frac{\kappa}{2}\frac{1}{N_t^2}\int_0^{N_t} x_{it}^2 di\right) - -\frac{\partial}{\partial N_t}\left(\frac{\tilde{k}}{2}\frac{1}{N_t}\int_0^{N_t} \tilde{x}_{it}^2 di\right)$$

but:

$$\frac{\partial}{\partial N_t}\left(\frac{\kappa}{2}\frac{1}{N_t^2}\int_0^{N_t} x_{it}^2 di\right) = \frac{\kappa}{2}\frac{\partial}{\partial N_t}\left[x_{it}^2 \frac{1}{N_t}\right] \quad = \frac{\kappa}{2}\left[-x_{it}^2 \frac{1}{N_t^2}\right]$$

and

$$\frac{\partial}{\partial N_t}\left(\frac{\tilde{k}}{2}\frac{1}{N_t}\int_0^{N_t} \tilde{x}_{it}^2 di\right) = \frac{\tilde{\kappa}}{2}\frac{\partial}{\partial N_t}\left[\tilde{x}_{it}^2\right] = 0$$

where the last equality results from symmetry. Thus:

$$\frac{\partial H}{\partial N_t} = \left[\frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}\right] + \frac{\kappa}{2}\left[x_{it}^2 \frac{1}{N_t^2}\right]$$

The FOC $\frac{\partial H}{\partial N_t} = (\rho - g_L)\mu_t - \dot{\mu}_t$ is then given by:

$$(\rho - g_L)\mu_t - \dot{\mu}_t = \frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} + \frac{\kappa}{2} x_{it}^2 \frac{1}{N_t^2}$$

A.3 Solving the Optimal Allocation

The 4 FOCs are:

$$\begin{cases} \dfrac{L_{pt}^{-1}}{1-\eta} = \dfrac{\mu_t}{\chi} & (A) \\ \dfrac{\eta}{1-\eta}\left(\dfrac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}\alpha = \kappa x_{it} & (B) \\ \dfrac{\eta}{1-\eta}\left(\dfrac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}(1-\alpha) = \tilde{\kappa}\tilde{x}_{it} & (C) \\ (\rho - g_L)\mu_t - \dot{\mu}_t = \dfrac{1}{(\sigma-1)\,N_t} - \dfrac{\alpha x_{it}}{N_t^2}\left(\dfrac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} + \dfrac{\kappa}{2}x_{it}^2\dfrac{1}{N_t^2} & (D) \end{cases}$$

Solving for $\tilde{x}_{it}$ and x_{it} (eqs. 24 and 25).

Divide (B) by (C):

$$\frac{\alpha}{(1-\alpha)} = \frac{\kappa}{\tilde{\kappa}}\left(\frac{x_{it}}{\tilde{x_{it}}}\right)$$

so

$$x_{it} = \frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{it}$$

Replace in (C):

$$\frac{\eta}{1-\eta}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}(1-\alpha) = \tilde{\kappa}\tilde{x}_{it}$$

$$\implies \frac{\eta}{1-\eta}\left(\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}(1-\alpha) = \tilde{\kappa}\tilde{x}_{it}$$

Solve for $\tilde{x}_{it}$:

$$\tilde{x}_{it}^2 = \left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)\left[\frac{(1-\alpha)}{\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)}{N_t} + (1-\alpha)}\right]$$

and $\lim\limits_{N_t\to\infty}\left[\frac{(1-\alpha)}{\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)}{N_t}+(1-\alpha)}\right] = 1$. Hence the optimal values for $\tilde{x}_{it}$ and x_{it} as N_t

grows large are:

$$\tilde{x}_{it}^{sp} = \left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)^{\frac{1}{2}} \tag{24}$$

$$x_{it}^{sp} = \frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)\left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)^{\frac{1}{2}}. \tag{25}$$

Solving for L_{it}^{sp} (eq. 26).

Using (A):

$$g_{L_{pt}} = -\frac{\dot{\mu}}{\mu}$$

Using this and (A) in (D):

$$(\rho - g_L) - \frac{\dot{\mu}_t}{\mu} = \frac{1}{\mu}\left[\frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1}\frac{\kappa}{2}x_{it}^2\frac{1}{N_t^2}\right]$$

$$\implies (\rho - g_L) + g_{L_{pt}} = \frac{L_{pt}(1-\eta)}{\chi}\left[\frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} + \frac{\kappa}{2}x_{it}^2\frac{1}{N_t^2}\right]$$

but $g_L = g_{L_p}$ [11]therefore:

$$\implies \rho = \frac{L_{pt}(1-\eta)}{\chi}\left[\frac{1}{(\sigma-1)N_t} - \frac{\alpha x_{it}}{N_t^2}\left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{-1} + \frac{\kappa}{2}x_{it}^2\frac{1}{N_t^2}\right]$$

use the approximation $\frac{1}{N_t^2} \simeq 0$ when N_t grows large, then:

$$L_{pt} = \frac{\rho\chi(\sigma-1)}{1-\eta}N_t$$

and also:

[11]Since $\dot{N}_t = \frac{1}{\chi}L_{et}$, then because of BGP $g_N = \frac{1}{\chi}\frac{L_{et}}{N_t}$, and therefore $g_{Le_t} = g_N$ since the RHS must be constant in time. On the other hand, $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$, hence $g_N = \frac{1}{\chi}\left(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t}\right)$,and since the RHS must be constant then $g_L = g_{L_{pt}} = g_N$.

$$g_N = g_{L_p}$$

Now use $L_{it} = \frac{L_{pt}}{N_t}$ and therefore:

$$L_{it}^{sp} = \frac{\rho\chi\left(\sigma - 1\right)}{1 - \eta} := \nu_{sp} \tag{26}$$

Solving for N_t^{sp} (eq. 27).

From the definition of the innovation process, $\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$ and therefore

$$\frac{L_t}{N_t} = \frac{\dot{N}_t}{N_t}\chi + \frac{L_{pt}}{N_t}$$

but from above $\frac{\dot{N}_t}{N_t} = g_N = g_{Lp}$ and $g_{L_p} = g_L$ therefore $\frac{\dot{N}_t}{N_t} = g_L$ and we get:

$$\frac{L_t}{N_t} = g_L\chi + \frac{L_{pt}}{N_t}$$

and from above we know that when N_t grows large, then $L_{pt} = \frac{\rho\chi(\sigma-1)}{1-\eta}N_t$, or also: $\frac{L_{pt}}{N_t} = \frac{\rho\chi(\sigma-1)}{1-\eta}$. Thus when N_t grows large we have:

$$\frac{L_t}{N_t} = g_L\chi + \frac{\rho\chi\left(\sigma - 1\right)}{1 - \eta}$$

or also:

$$N_t = L_t\left[\frac{1}{g_L\chi + \frac{\rho\chi(\sigma-1)}{1-\eta}}\right]$$

but $\frac{\rho\chi(\sigma-1)}{1-\eta} = \nu_{sp}$ and therefore we finally get:

$$N_t^{sp} = \frac{L_t}{\chi g_L + \nu_{sp}} = \psi_{sp}L_t \tag{27}$$

Solving for L_{pt}^{sp} (eq. 28).

Use $L_{pt} = \frac{\rho\chi(\sigma-1)}{1-\eta}N_t$ and equation (25) $N_t^{sp} = \frac{L_t}{\chi g_L + \nu_{sp}} = \psi_{sp}L_t$ to get :

$$\begin{aligned} L_{pt}^{sp} &= \frac{\rho\chi(\sigma-1)}{1-\eta} N_t \\ &= \frac{\rho\chi(\sigma-1)}{1-\eta} \psi_{sp} L_t \\ &= \nu_{sp}\psi_{sp} L_t \end{aligned} \tag{28}$$

Solving for Y_t^{sp} (eq. 29).

Aggregate production is given by $Y_t = N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. Use the optimal expression for N_t^{sp}, L_{pt}^{sp}, $\tilde{x}_{sp}$ and x_{sp} to get:

$$\begin{aligned} Y_t &= N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x_{it}}\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}. \\ &= (\psi_{sp}L_t)^{\frac{1}{\sigma-1}} \left(\alpha\frac{\alpha}{N_t}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{sp} + (1-\alpha)\tilde{x_{sp}}\right)^{\frac{\eta}{1-\eta}} (\nu_{sp}\psi_{sp}L_t)^{\frac{1}{1-\eta}} \\ &= [v_{sp}]^{\frac{1}{1-\eta}} \left(\alpha\frac{\alpha}{N_t}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{sp} + (1-\alpha)\tilde{x_{sp}}\right)^{\frac{\eta}{1-\eta}} (\psi_{sp}L_t)^{\frac{1}{1-\eta}+\frac{1}{\sigma-1}} \end{aligned}$$

Assuming $\alpha\frac{\alpha}{N_t}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{sp} \simeq 0$ when N_t is high, then:

$$Y_t = [v_{sp}(1-\alpha)^\eta \tilde{x_{sp}}^\eta]^{\frac{1}{1-\eta}} (\psi_{sp}L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} \tag{29}$$

Solving for c_t^{sp} (eq. 30).

By definition, consumption per capita is:

$$\begin{aligned} c_t &= \frac{Y_t}{L_t} \\ &= \frac{[v_{sp}(1-\alpha)^\eta \tilde{x_{sp}}^\eta]^{\frac{1}{1-\eta}} (\psi_{sp}L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}}}{L_t} \\ &= [v_{sp}(1-\alpha)^\eta \tilde{x_{sp}}^\eta]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}} \end{aligned} \tag{30}$$

Solving for g_c^{sp} (eq. 31).

Using the definition of consumption per capita growth and equation 30 yields:

$$g_c^{sp} = g_{Y/L}^{sp} = (\frac{1}{\sigma - 1} + \frac{\eta}{1-\eta})g_L \tag{31}$$

Solving for D_{it}^{sp} (eq. 32).

From equation 19, $Y_t = N_t^{\frac{1}{\sigma-1}} D_{it}^{\eta} L_{pt}$. Plugging the previous results yields:

$$\begin{aligned} D_{it}^{sp} &= \left(\frac{Y_t}{N_t^{\frac{1}{\sigma-1}} L_{pt}} \right)^{\frac{1}{\eta}} \\ &= \left(\frac{[\upsilon_{sp}(1-\alpha)^{\eta} \tilde{x_{sp}}^{\eta}]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}}}{(\psi_{sp} L_t)^{\frac{1}{\sigma-1}} \nu_{sp} \psi_{sp} L_t} \right)^{\frac{1}{\eta}} \\ &= \left([(1-\alpha)\tilde{x_{sp}}]^{\frac{\eta}{1-\eta}} (\upsilon_{sp} \psi_{sp} L_t)^{\frac{\eta}{1-\eta}} \right)^{\frac{1}{\eta}} \\ &= [(1-\alpha)\tilde{x_{sp}} \upsilon_{sp} \psi_{sp} L_t]^{\frac{1}{1-\eta}} \end{aligned} \tag{32}$$

Solving for D_t^{sp} (eq. 33).

By definition:

$$\begin{aligned} D_t^{sp} &= N_t^{sp} D_{it}^{sp} \\ &= (\psi_{sp} L_t) \left([(1-\alpha)\tilde{x_{sp}} \upsilon_{sp} \psi_{sp} L_t]^{\frac{1}{1-\eta}} \right) \\ &= [(1-\alpha)\tilde{x_{sp}} \upsilon_{sp}]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{1-\eta}+1} \end{aligned} \tag{33}$$

Solving for Y_{it}^{sp} (eq. 34).

Using equation 18 and the results for L_{pt} and D_{it}:

$$\begin{aligned} Y_{it}^{sp} &= D_{it}^{\eta} \frac{L_{pt}}{N_t} \\ &= [(1-\alpha)\tilde{x_{sp}} \upsilon_{sp} \psi_{sp} L_t]^{\frac{\eta}{1-\eta}} (\nu_{sp}) \\ &= \left[(1-\alpha)^{\eta} \tilde{x_{sp}^{\eta}} \upsilon_{sp} \right]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{\eta}{1-\eta}} \end{aligned} \tag{34}$$

Solving for Y_0^{sp} (eq. 35).

By definition $U_0 = \int_0^\infty e^{-(\rho-g_L)} L_0 u(c_t, x_{it}, \tilde{x}_{it})dt$. Then:

$$U_0 = \int_0^\infty e^{-(\rho-g_L)t} L_0 u(c_t, x_{it}, \tilde{x}_{it})dt$$
$$= L_0 \int_0^\infty e^{-(\tilde{\rho})t} \left[log([v_{sp}(1-\alpha)^\eta \tilde{x_{sp}}^\eta]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}})... \right.$$
$$\left. ... - \frac{\kappa}{2N_t} x_{sp}^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \right]$$

Next, assume $\frac{\kappa}{2N_t} x_{sp}^2 \simeq 0$ and use $c_0 = [v_{sp}(1-\alpha)^\eta \tilde{x_{sp}}^\eta]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_0^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$. Then:

$$U_0 = L_0 \int_0^\infty e^{-(\tilde{\rho})t} \left[log(c_0) + \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}\right) g_L t \right] dt + ...$$
$$- L_0 \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \int_0^\infty e^{-(\tilde{\rho})t} dt$$
$$= L_0 \frac{1}{\tilde{\rho}} \left(log(c_0) - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \right) + L_0 \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}\right) g_L \int_0^\infty e^{-(\tilde{\rho})t} t dt$$
$$= L_0 \frac{1}{\tilde{\rho}} \left(log(c_0) - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \right) + L_0 \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}\right) g_L \frac{1}{\tilde{\rho}^2}$$
$$= L_0 \frac{1}{\tilde{\rho}} \left(log(c_0) - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 + \frac{g_c}{\tilde{\rho}} \right) \quad (35)$$

B Competitive Equilibrium When Firms Own Data

B.1 Household Problem

The problem is defined by:

$$U_0 = \max_{\{c_{it}\}} \int_0^\infty e^{-(\tilde{\rho})} L_0 u(c_t, x_{it}, \tilde{x}_{it})dt$$
$$s.t.\ c_t = \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$
$$\dot{a}_t = (r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it} c_{it} di$$

Define the current value Hamiltonian with state variable a_t, control variable c_{it} and co-state variable μ_t:

$$H(a_t, c_{it}, \mu_t) = u(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it}) + \mu_t \left[(r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it} c_{it} di\right]$$

The FOCs are:

$$\begin{cases} \dfrac{\partial H}{\partial c_{it}} = 0 \\ \dfrac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t \end{cases}$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma-1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\sigma-1} c_{it} = (\mu_t p_{it})^{-\sigma}$$

$$\Rightarrow \; \left(c_t^{\sigma-1} c_{it}\right)^{\frac{\sigma-1}{\sigma}} = (\mu_t p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{\sigma-1}{\sigma} + \frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \ \mu_t = \frac{1}{c_t \left(\int_0^{N_t} p_{it}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}}$$

Next, use the fact that the price of c_t is normalized to 1, i.e., $P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} = 1$. Next, plug the expression for μ_t in the FOC and this gives equation (39):

$$c_{it} = c_t p_{it}^{-\sigma}$$

Next, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t(r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} + g_c$$

B.2 Firm Problem

The firm problem is:

$$r_t V_{it} = \underset{\{L_{it}, D_{bit}, x_{it}, \tilde{x}_{it}\}}{max} \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit} \tilde{x}_{it} Y_{it} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it}$$

$$s.t. \ Y_{it} = D_{it}^{\eta} L_{it}$$

$$D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit}$$

$$p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{B_t}{\tilde{x}_{it} Y_{it}} \right)^{\frac{1}{\epsilon}}$$

$$x_{it} \in [0;1]$$

$$\tilde{x}_{it} \in [0;1]$$

taking as given λ_{DI}, B_t, N_t, p_{bt} and Y_t. To solve this problem, write the Lagrangean:

$$\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} - p_{bt} D_{bit} + p_{sit} \tilde{x}_{it} Y_{it} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it}....$$
$$+\mu_{x0}(x_{it}) + \mu_{\tilde{x}0}(\tilde{x}_{it}) + \mu_{\tilde{x}1}(1-\tilde{x}_{it}) + \mu_{x1}(1-x_{it}) + \mu_{0d} D_{bit}$$

Simplify using the constraints:

$$\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} - p_{bt} D_{bit} + \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it} Y_{it})^{1-\frac{1}{\epsilon}} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it}....$$
$$+\mu_{x0}(x_{it}) + \mu_{\tilde{x}0}(\tilde{x}_{it}) + \mu_{\tilde{x}1}(1-\tilde{x}_{it}) + \mu_{x1}(1-x_{it})$$

Now take the FOCs.

1) Start with the FOC w.r.t. to L_{it}:

$$\frac{\partial \mathbb{L}}{\partial L_{it}} = 0$$
$$\Leftrightarrow\ (1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial L_{it}} - w_t + (1-\frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it})^{1-\frac{1}{\epsilon}} \frac{\partial Y_{it}}{\partial L_{it}} = 0$$
$$\frac{\partial Y_{it}}{\partial L_{it}} \left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = w_t$$

And using $Y_{it} = D_{it}^{\eta} L_{it}$ and assuming D_{it} depends on L_{it}, then by implicit derivation:

$$\frac{\partial Y_{it}}{\partial L_{it}} = \eta D_{it}^{\eta-1} L_{it} \frac{\partial D_{it}}{\partial L_{it}} + D_{it}^{\eta}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

Next, compute $\frac{\partial D_{it}}{\partial L_{it}}$ using $D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit}$:

$$\frac{\partial D_{it}}{\partial L_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}}$$

Substituting above:

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}$$

The FOC for L_{it} is then:

$$\frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t$$

2) Compute the FOC w.r.t. to D_{bit}:

$$\frac{\partial \mathbb{L}}{\partial D_{bit}}=0$$

$$\Leftrightarrow (1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}\frac{\partial Y_{it}}{\partial D_{bit}}-w_t+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}\left(B_t\right)^{\frac{1}{\epsilon}}\left(\tilde{x}_{it}\right)^{1-\frac{1}{\epsilon}}\frac{\partial Y_{it}}{\partial D_{bit}}-p_{bt}+\mu_{0d}=0$$

$$\frac{\partial Y_{it}}{\partial D_{bit}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=p_{bt}$$

And using $Y_{it}=D_{it}^{\eta}L_{it}$ we have:

$$\frac{\partial Y_{it}}{\partial D_{bit}}=\eta\frac{Y_{it}}{D_{it}}\frac{\partial D_{it}}{\partial D_{bit}}$$

and using $D_{it}=\alpha x_{it}Y_{it}+(1-\alpha)D_{bit}$:

$$\frac{\partial D_{it}}{\partial D_{bit}}=\alpha x_{it}\frac{\partial Y_{it}}{\partial D_{bit}}+(1-\alpha)$$

Substituting above:

$$\frac{\partial Y_{it}}{\partial D_{bit}}=\eta\frac{Y_{it}}{D_{it}}\left(\alpha x_{it}\frac{\partial Y_{it}}{\partial D_{bit}}+(1-\alpha)\right)$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial D_{bit}}=\frac{(1-\alpha)\eta\frac{Y_{it}}{D_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}$$

Then the FOC for D_{bit} is

$$\frac{(1-\alpha)\eta\frac{Y_{it}}{D_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=p_{bt}$$

3) Compute the FOC w.r.t. to x_{it}:

$$\frac{\partial\mathbb{L}}{\partial x_{it}}=0$$

$$\Leftrightarrow (1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}\frac{\partial Y_{it}}{\partial x_{it}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}(B_t)^{\frac{1}{\epsilon}}(\tilde{x}_{it})^{1-\frac{1}{\epsilon}}\frac{\partial Y_{it}}{\partial x_{it}}+\mu_{x0}-\mu_{x1}=0$$

$$\frac{\partial Y_{it}}{\partial x_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=-\mu_{x0}+\mu_{x1}$$

Now to compute $\frac{\partial Y_{it}}{\partial x_{it}}$ use $Y_{it}=D_{it}^{\eta}L_{it}$:

$$\frac{\partial Y_{it}}{\partial x_{it}}=\eta\frac{Y_{it}}{D_{it}}\frac{\partial D_{it}}{\partial x_{it}}$$

and using $D_{it}=\alpha x_{it}Y_{it}+(1-\alpha)D_{bit}$ and implicit derivation:

$$\frac{\partial D_{it}}{\partial x_{it}}=\alpha x_{it}\frac{\partial Y_{it}}{\partial x_{it}}+Y_{it}\alpha$$

Thus:

$$\Rightarrow\frac{\partial Y_{it}}{\partial x_{it}}=\eta\frac{Y_{it}}{D_{it}}\left[\alpha x_{it}\frac{\partial Y_{it}}{\partial x_{it}}+Y_{it}\alpha\right]$$

$$\Rightarrow\frac{\partial Y_{it}}{\partial x_{it}}=\frac{\eta\frac{Y_{it}}{D_{it}}Y_{it}\alpha}{1-\alpha x_{it}\eta\frac{Y_{it}}{D_{it}}}$$

Then the FOC w.r.t. to x_{it} is:

$$\frac{\eta\frac{Y_{it}}{D_{it}}Y_{it}\alpha}{1-\alpha x_{it}\eta\frac{Y_{it}}{D_{it}}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=-\mu_{x0}+\mu_{x1}$$

Now, note that the LHS is >0, then:

$$\mu_{x1} > \mu_{x0} \geq 0$$

$$\Rightarrow \mu_{x1} > 0$$

$$\Rightarrow x_{it} = 1 \tag{A.1}$$

4) Compute the FOC w.r.t. to $\tilde{x}_{it}$:

$$\frac{\partial \mathbb{L}}{\partial \tilde{x}_{it}} = 0$$

$$\Leftrightarrow \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (Y_{it})^{1-\frac{1}{\epsilon}} \tilde{x}_{it}^{-\frac{1}{\epsilon}} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} + \mu_{\tilde{x}0} - \mu_{\tilde{x}1} = 0$$

$$\lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (Y_{it})^{1-\frac{1}{\epsilon}} \tilde{x}_{it}^{-\frac{1}{\epsilon}} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} - \mu_{\tilde{x}0} + \mu_{\tilde{x}1}$$

Or also:

$$p_{sit} Y_{it} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} = -\mu_{\tilde{x}0} + \mu_{\tilde{x}1}$$

Finally, the 4 FOCs of the firm problem are:

$$\begin{cases} \frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = w_t & (A) \\ \frac{(1-\alpha)\eta \frac{Y_{it}}{D_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = p_{bt} & (B) \\ x_{it} = 1 & (C) \\ p_{sit} Y_{it} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} + \mu_{\tilde{x}0} - \mu_{\tilde{x}1} = 0 & (D) \end{cases}$$

5) Solution in terms of equilibrium aggregates

Divide (A) by (B) :

$$\left[\frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\right]\left[\frac{(1-\alpha)\eta\frac{Y_{it}}{D_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\right]^{-1} = \frac{w_t}{p_{bt}}$$

$$\Rightarrow \frac{D_{it}}{(1-\alpha)\eta L_{it}} = \frac{w_t}{p_{bt}}$$

Hence:

$$\Rightarrow D_{it} = \frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it} \tag{A.2}$$

Next, substitute in (A) and use (C):

$$\frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right] = w_t$$

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right] = w_t\frac{L_{it}}{Y_{it}}\left(1-\eta\frac{Y_{it}}{D_{it}}\alpha\right)$$

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right] = w_t\frac{L_{it}}{Y_{it}} - w_t\eta\frac{L_{it}}{D_{it}}\alpha$$

But from above $D_{it} = \frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it}$, therefore substitute in RHS:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right] = w_t\frac{L_{it}}{Y_{it}} - \alpha(\frac{p_{bt}}{1-\alpha})$$

Then:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}} + \alpha(\frac{p_{bt}}{1-\alpha})\right]\frac{Y_{it}}{L_{it}} = w_t \tag{A.3}$$

Next, we need to compute $\frac{Y_{it}}{L_{it}}$ using: $Y_{it} = D_{it}^{\eta}L_{it}$:

$$\frac{Y_{it}}{L_{it}} = D_{it}^{\eta} = \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta}L_{it}^{\eta} \tag{A.4}$$

B.3 Data Intermediary Problem

The problem faced by the data intermediary is :

$$\max_{p_{bt},D_{sit}} p_{bt}\int_0^{N_t} D_{bit}di - \int_0^{N_t} p_{sit}D_{sit}di$$
$$s.t. D_{bit} \leq B_t = \left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}$$
$$p_{bt} \leq p_{bt}^*$$

where p_{sit}, i.e. the purchase price of data is taken as given.

B.3.1 The downward-sloping demand curve: data intermediary cost minimization

To compute the demand curve of the data intermediary, we solve the following cost minimization problem:

$$\min_{D_{sit}} \int_0^{N_t} p_{sit}D_{sit}di$$
$$s.t.\ D_{bit} \leq B_t = \left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}$$

The Lagrangean is given by:

$$\mathbb{L} = \int_0^{N_t} p_{sit}D_{sit}di + \lambda_{DI}\left[D_{bit} - \left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}\right]$$

By symmetry:

$$\mathbb{L} = \int_0^{N_t} p_{sit}D_{sit}di + \lambda_{DI}\left[D_{bit} - \left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}\right]$$

Taking FOC yields:

$$\frac{\partial \mathbb{L}}{\partial D_{sit}} = 0$$
$$\Leftrightarrow\ p_{sit} - \lambda_{DI}\frac{\epsilon}{\epsilon-1}\left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}-1}\frac{\epsilon-1}{\epsilon}D_{sit}{}^{\frac{\epsilon-1}{\epsilon}-1} = 0$$
$$p_{sit} - \lambda_{DI}B_t^{\frac{1}{\epsilon}}D_{sit}{}^{\frac{\epsilon-1}{\epsilon}-1}N_t^{-\frac{1}{\epsilon}} = 0$$
$$\lambda_{DI}\left(\frac{B_t}{D_{sit}}\right)^{\frac{1}{\epsilon}}N_t^{-\frac{1}{\epsilon}} = p_{sit}$$

We get the following demand curve:

$$p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{B_t}{D_{sit}} \right)^{\frac{1}{\epsilon}}$$

which is equation (44), which is used as a constraint in the firm problem. Next, by symmetry

$$B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} (D_{sit})^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t^{\frac{1}{-\epsilon}\frac{\epsilon}{\epsilon-1} + \frac{\epsilon}{\epsilon-1}} D_{sit} = N_t D_{sit} \tag{A.5}$$

Substituting in equation (44)

$$p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{N_t D_{sit}}{D_{sit}} \right)^{\frac{1}{\epsilon}}$$

$$\Rightarrow p_{sit} = \lambda_{DI}$$

B.3.2 The zero profit condition

The objective function is increasing in p_{bt}. It is clear that $p_{bt} = p^*_{bt}$, which is the price given by the zero profit condition.Use the zero profit condition:

$$\begin{aligned}
\Pi &= 0 \\
p_{bt} \int_0^{N_t} D_{bit} di - \int_0^{N_t} p_{sit} D_{sit} di &= 0 \\
p_{bt} \int_0^{N_t} B_t di - \int_0^{N_t} p_{sit} D_{sit} di &= 0 \\
p_{bt} B_t N_t - p_{sit} D_{sit} N_t &= 0
\end{aligned}$$

Thus we get:

$$p_{bt} = \frac{p_{sit} D_{sit}}{B_t}$$

Next, from the demand curve, we have $\lambda_{DI} \left(\frac{B_t}{D_{sit}} \right)^{\frac{1}{\epsilon}} N_t^{-\frac{1}{\epsilon}} = p_{sit}$ and solving for B_t we get $B_t = \left(\frac{p_{sit}}{\lambda_{DI}} \right)^{\epsilon} N_t D_{sit} = N_t D_{sit}$ where the last equation comes from the fact that $p_{sit} = \lambda_{DI}$. Finally:

$$p_{bt} = \frac{p_{sit}D_{sit}}{N_t D_{sit}}$$

$$\Rightarrow p_{bt} = \frac{p_{sit}}{N_t} \tag{A.6}$$

B.4 Free Entry and the Creation of New Varieties

The free entry condition is given by:

$$\chi w_t = V_{it} + \frac{\int_0^{N_t} \delta(\tilde{x}_{it})V_{it}di}{\dot{N}_t}$$

by symmetry:

$$\chi w_t = V_{it} + \frac{\delta(\tilde{x}_{it})V_{it}}{\frac{\dot{N}_t}{N_t}}$$

$$\chi w_t = V_{it}(1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

B.5 Equilibrium when Firms Own Data

B.5.1 Solve for p_{bt}, Y_{it}, D_{sit}, D_{bit}, w_t as a function of $\tilde{x}_{it}$ and aggregates

Take the FOC w.r.t. to L_i in the firm problem, i.e. A.3:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}} + \alpha(\frac{p_{bt}}{1-\alpha})\right]\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta} L_{it}^{\eta} = w_t$$

Now, from the date intermediary problem, i.e. A.6, $p_{bt} = \frac{p_{sit}}{N_t} = \lambda_{DI}N_t^{-\frac{1}{\epsilon}-1}\left(\frac{B_t}{\tilde{x}_{it}Y_{it}}\right)^{\frac{1}{\epsilon}}$. Substitute above:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})\tilde{x}_{it}p_{bt}N_t + \alpha(\frac{p_{bt}}{1-\alpha})\right]\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta} L_{it}^{\eta} = w_t$$

Next, from A.2, $D_{it} = \frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it}$, then

$$p_{bt} = \frac{w_t}{D_{it}}(1-\alpha)\eta L_{it}$$

Use (A.4) to get $\frac{Y_{it}}{D_{it}} = D_{it}^{\eta-1} L_{it} = \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1} L_{it}^{\eta}$. Substitute:

$$\begin{aligned} p_{bt} &= \frac{w_t}{D_{it}}(1-\alpha)\eta L_{it} \\ &= \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1} L_{it}^{\eta} w_t(1-\alpha)\eta \left(\frac{Y_{it}}{L_{it}}\right)^{-1} \end{aligned}$$

Next, from equation (13), by symmetry, we know that $L_{it} = \frac{L_{pt}}{N_t}$. Moreover, by symmetry, $c_t = N_t^{\frac{\sigma}{\sigma-1}} c_{it} = N_t^{\frac{\sigma}{\sigma-1}} \frac{Y_{it}}{L_t}$ and therefore $Y_t = c_t L_t = N_t^{\frac{\sigma}{\sigma-1}} \frac{Y_{it}}{L_t} L_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it}$. Thus $\frac{Y_{it}}{L_{it}} = \left(\frac{Y_t}{N_t^{\frac{\sigma}{\sigma-1}}} \frac{N_t}{L_{pt}}\right) = N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}}$. Substituting above:

$$\begin{aligned} p_{bt} &= \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1} L_{it}^{\eta} w_t(1-\alpha)\eta \left(\frac{Y_{it}}{L_{it}}\right)^{-1} \\ p_{bt} &= \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1} \left(\frac{L_{pt}}{N_t}\right)^{\eta} w_t(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}} \frac{L_{pt}}{Y_t} \\ p_{bt}^{\eta} &= \left[w_t(1-\alpha)\eta\right]^{\eta} L_{pt}^{\eta+1} N_t^{\frac{1}{\sigma-1}-\eta} \frac{1}{Y_t} \\ p_{bt} &= w_t(1-\alpha)\eta L_{pt} N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}} \left(\frac{L_{pt}}{Y_t}\right)^{\frac{1}{\eta}} \\ p_{bt} &= (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} \end{aligned} \tag{A.7}$$

Now, we need to solve Y_t. For that, from above $\frac{Y_{it}}{L_{it}} = N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}}$ and $L_{it} = \frac{L_{pt}}{N_t}$. Thus:

$$Y_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it} \tag{A.8}$$

But from the firm's problem we have

$$Y_{it} = D_{it}^{\eta} L_{it} \tag{A.9}$$

Therefore, solve for D_{it}, where:

$$D_{it} = \alpha Y_{it} + (1-\alpha) D_{bit} \tag{A.10}$$

But from the data intermediary problem $D_{bit} = B_t$ and from A.5, $D_{bit} = B_t = N_t D_{sit}$, and by definition $D_{sit} = \tilde{x}_{it} Y_{it}$. Thus:

$$D_{bit} = N_t \tilde{x}_{it} Y_{it} \tag{A.11}$$

Substitute A.11 in A.10and we get:

$$\begin{aligned} D_{it} &= \alpha Y_{it} + (1-\alpha) D_{bit} \\ &= \alpha Y_{it} + (1-\alpha) N_t \tilde{x}_{it} Y_{it} \\ &= Y_{it}(\alpha + (1-\alpha) N_t \tilde{x}_{it}) \end{aligned} \tag{A.12}$$

Substitute in A.9:

$$\begin{aligned} Y_{it} &= \left(Y_{it}(\alpha + (1-\alpha) N_t \tilde{x}_{it})\right)^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\eta} \left(\frac{L_{pt}}{N_t}\right) \end{aligned}$$

$$\Rightarrow Y_{it} = (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \tag{A.13}$$

Substitute this expression for Y_{it} in A.8:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \end{aligned} \tag{A.14}$$

Use this in A.7:

$$\begin{aligned} p_{bt} &= (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1} - \eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} \\ &= (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1} - \eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} \left(N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}\right)^{1-\frac{1}{\eta}} \\ &= (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1} - \eta\right)\frac{1}{\eta} + \left(\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}\right)\left(1-\frac{1}{\eta}\right)} L_{pt}^{\frac{1}{\eta} + \frac{1}{1-\eta}\left(1-\frac{1}{\eta}\right)} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\left(\frac{\eta}{1-\eta}\right)\left(1-\frac{1}{\eta}\right)} \\ &= (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}} L_{pt}^{0} \left(\alpha + (1-\alpha) \tilde{x}_{it} N_t\right)^{-1} \end{aligned} \tag{A.15}$$

Finally,

$$p_{bt} = (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}} (\alpha + (1-\alpha)\tilde{x_{it}} N_t)^{-1} \tag{A.16}$$

We will use this in the FOC with respect to L_{it}. Recall from A.3:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} + \alpha(\frac{p_{bt}}{1-\alpha})\right]\left(\frac{Y_{it}}{L_{it}}\right) = w_t$$

But from A.6, $p_{bt} = \frac{p_{sit}}{N_t}$ and by the cost minimization problem of the data intermediary, $p_{bt} = \frac{p_{sit}}{N_t} = \frac{1}{N_t}\lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{B_t}{\tilde{x}_{it} Y_{it}}\right)^{\frac{1}{\epsilon}}$. Substituting above:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon}) p_{bt} \tilde{x}_{it} N_t + \alpha(\frac{p_{bt}}{1-\alpha})\right]\left(\frac{Y_{it}}{L_{it}}\right) = w_t$$

but from above $\frac{Y_{it}}{L_{it}} = N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}}$ and therefore:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon}) p_{bt} \tilde{x}_{it} N_t + \alpha(\frac{p_{bt}}{1-\alpha})\right] N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}} = w_t$$

Also, using A.8:

$$\Rightarrow (1-\frac{1}{\sigma}) N_t^{\frac{1}{\sigma-1}} + p_{bt}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it} N_t + \frac{\alpha}{1-\alpha}\right] = N_t^{\frac{1}{\sigma-1}} \left(\frac{w_t L_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma}) N_t^{\frac{1}{\sigma-1}} + (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}} (\alpha + (1-\alpha)\tilde{x_{it}} N_t)^{-1}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it} N_t + \frac{\alpha}{1-\alpha}\right] = N_t^{\frac{1}{\sigma-1}} \left(\frac{w_t L_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma}) + (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) (\alpha + (1-\alpha)\tilde{x_{it}} N_t)^{-1}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it} N_t + \frac{\alpha}{1-\alpha}\right] = \left(\frac{w_t L_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma}) + (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) \frac{(1-\frac{1}{\epsilon})\tilde{x}_{it} N_t + \frac{\alpha}{1-\alpha}}{\alpha + (1-\alpha)\tilde{x_{it}} N_t} = \left(\frac{w_t L_{pt}}{Y_t}\right)$$

Next, define $f(N_t) = \frac{(1-\frac{1}{\epsilon})\tilde{x}_{it} N_t + \frac{\alpha}{1-\alpha}}{\alpha + (1-\alpha)\tilde{x_{it}} N_t}$ Thus:

$$\Rightarrow (1-\frac{1}{\sigma}) + (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) f(N_t) = \left(\frac{w_t L_{pt}}{Y_t}\right)$$

$$\Rightarrow \frac{w_t L_{pt}}{Y_t} = \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} \tag{A.17}$$

When N_t is large, $\lim_{N_t\to\infty} f(N_t) = \frac{(1-\frac{1}{\epsilon})}{(1-\alpha)}$. Therefore:

$$\Rightarrow \left(\frac{w_t L_{pt}}{Y_t}\right) = \frac{(\sigma-1)}{\sigma(1-\eta\frac{\epsilon}{\epsilon-1})} \tag{A.18}$$

Next, substitute A.14 in A.17 :

$$\begin{aligned} w_t &= \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\frac{Y_t}{L_{pt}} \\ &= \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} \end{aligned} \tag{A.19}$$

So far we have:

$$\begin{cases} \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} = w_t \\ N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} = Y_t \\ (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{(\frac{1}{\sigma-1}-\eta)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} = p_{bt} \\ p_{bt} = \frac{p_{sit}}{N_t} \\ 1 = x_{it} \\ Y_{it}(\alpha+(1-\alpha)N_t\tilde{x}_{it}) = D_{it} \\ N_t\tilde{x}_{it}Y_{it} = D_{bit} \\ Y_t N_t^{-\frac{\sigma}{\sigma-1}} = Y_{it} \\ N_t D_{sit} = B_t \\ \tilde{x}_{it}Y_{it} = D_{sit} \\ L_{it} = \frac{L_{pt}}{N_t} \end{cases} \tag{A.20}$$

B.5.2 The Value of the Firm and Profits

The firm's problem is:

$$r_t V_{it} = \max_{\{L_{it}, D_{bit}, x_{it}, \tilde{x}_{it}\}} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit}\tilde{x}_{it} Y_{it} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it}$$

Thus:

$$V_{it} = \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit}\tilde{x}_{it} Y_{it}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}}$$

Next, define $\pi_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it}$. Then using the expressions above:

$$\begin{aligned}
\pi_{it} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it} \\
&= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w\left(\frac{L_{pt}}{N_t}\right) \\
&= \left(N_t^{\frac{1}{\sigma-1}}\right) N_t^{-\frac{\sigma}{\sigma-1}} Y_t - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} \left(\frac{L_{pt}}{N_t}\right) \\
&= N_t^{-1} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} \\
&= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}-1} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}-1} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \\
&= (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)
\end{aligned} \tag{A.21}$$

On the other hand:

$$\begin{aligned}
-p_{bt} D_{bit} + p_{sit}\tilde{x}_{it} Y_{it} &= -\left(\frac{p_{sit}}{N_t}\right)(N_t \tilde{x}_{it} Y_{it}) + p_{sit}\tilde{x}_{it} Y_{it} \\
&= 0
\end{aligned} \tag{A.22}$$

Hence, the value of the firm is given by:

$$
\begin{aligned}
V_{it} &= \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit}\tilde{x}_{it} Y_{it}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right) \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \qquad \text{(A.23)} \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(\frac{\sigma(1-(1-\alpha)\eta f(N_t)) - (\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(\frac{1-\sigma(1-\alpha)\eta f(N_t)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right)
\end{aligned}
$$

B.5.3 The Free Entry Condition

$$\chi w_t = V_{it}\left(1 + \frac{\delta(\tilde{x}_{it})}{g_L}\right)$$

Using A.19 and A.23:

$$\chi \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} = \dots$$

$$\dots \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)\left(1 + \frac{\delta(\tilde{x}_{it})}{g_L}\right)$$

$$\Rightarrow \chi \frac{(1-\frac{1}{\sigma})N_t}{1-(1-\alpha)\eta f(N_t)} L_{pt}^{-1} = \frac{1}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right) \left(1 + \frac{\delta(\tilde{x}_{it})}{g_L}\right)$$

$$\left(\frac{N_t}{L_{pt}}\right) = \frac{(1-(1-\alpha)\eta f(N_t))\,\sigma}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)} \left(1 - \frac{\sigma-1}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \left(1 + \frac{\delta(\tilde{x}_{it})}{g_L}\right)$$

$$\Rightarrow \left(\frac{N_t}{L_{pt}}\right) = \frac{\sigma(1-(1-\alpha)\eta f(N_t)) - \sigma + 1}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma - 1)} \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \left(\frac{N_t}{L_{pt}}\right) = \frac{1 - \sigma(1-\alpha)\eta f(N_t)}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma - 1)} \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \left(\frac{L_{pt}}{N_t}\right) = \frac{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma - 1)}{1 - \sigma(1-\alpha)\eta f(N_t)} \quad (\frac{g_L}{g_L + \delta(\tilde{x}_{it})}) \tag{A.24}$$

From the evolution of the number of varieties, we have $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

In BGP: $\frac{\dot{N}_t}{N_t} = g_N$ and since $g_N = g_L$, then:

$$\Rightarrow g_L = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

$$\Rightarrow \frac{L_t}{N_t} = \frac{L_{pt}}{N_t} + \chi g_L$$

Next, using $\frac{\dot{V}_{it}}{V_{it}} = g_\pi$ and substituting A.24:

$$\Rightarrow \frac{L_t}{N_t} = \frac{\chi\left(r + \delta(\tilde{x}_{it}) - g_\pi\right)(\sigma - 1)}{1 - \sigma(1-\alpha)\eta f(N_t)}(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}) + \chi g_L$$

$$\Rightarrow N_t = \frac{L_t}{\frac{\chi(r+\delta(\tilde{x}_{it})-g_\pi)(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}(\frac{g_L}{g_L+\delta(\tilde{x}_{it})}) + \chi g_L}$$

Define $\nu(N_t) = \frac{\chi(r+\delta(\tilde{x}_{it})-g_\pi)(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}(\frac{g_L}{g_L+\delta(\tilde{x}_{it})})$, so that:

$$N_t = \frac{L_t}{\nu(N_t) + \chi g_L}$$

Moreover, define $\psi(N_t) = \frac{1}{\nu(N_t)+\chi g_L}$ so that:

$$N_t = L_t \psi(N_t) \tag{A.25}$$

B.6 Solution of the Competitive Equilibrium

B.6.1 Data shared with other firms

From above, the FOC for $\tilde{x}_{it}$ is:

$$p_{sit}Y_{it}\left(1-\frac{1}{\epsilon}\right)-\delta'(\tilde{x_{it}})V_{it}+\mu_{\tilde{x}0}-\mu_{\tilde{x}1}=0$$

Assume an interior solution so $\mu_{\tilde{x}0}=\mu_{\tilde{x}1}=0$.

$$p_{sit}Y_{it}\left(1-\frac{1}{\epsilon}\right)-\delta'(\tilde{x_{it}})V_{it}=0$$

but from above $p_{bt}=\frac{p_{sit}}{N_t}$ and $p_{bt}=(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)N_t^{\frac{1}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}$ Therefore:

$$(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}Y_{it}\frac{(\epsilon-1)}{\epsilon}-\delta'(\tilde{x_{it}})V_{it}=0$$

Substitute $\frac{w_tL_{pt}}{Y_t}$ using A.17:

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}Y_{it}\frac{(\epsilon-1)}{\epsilon}-\delta'(\tilde{x_{it}})V_{it}=0$$

Substitute Y_{it} using A.13:

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}\frac{(\epsilon-1)}{\epsilon}=\delta'(\tilde{x_{it}})V_{it}$$

Next, substitute V_{it} using

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}\frac{(\epsilon-1)}{\epsilon}=\delta'(\tilde{x_{it}})\frac{\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r+\delta(\tilde{x}_{it})-\frac{\dot{V}_{it}}{V_{it}}}\left(\frac{1-\sigma(1-\alpha)\eta f(N_t)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right)$$

Simplify:

$$(1-\alpha)\eta(\sigma-1)N_t(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{-1}\frac{(\epsilon-1)}{\epsilon}=\delta'(\tilde{x_{it}})\frac{1-\sigma(1-\alpha)\eta f(N_t)}{r+\delta(\tilde{x}_{it})-g_\pi}$$

$$\Rightarrow\frac{\delta'(\tilde{x_{it}})}{r+\delta(\tilde{x}_{it})-g_\pi}(\alpha+(1-\alpha)N_t\tilde{x}_{it})=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta N_t\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow\frac{\delta'(\tilde{x_{it}})}{r+\delta(\tilde{x}_{it})-g_\pi}(\frac{\alpha}{N_t}+(1-\alpha)\tilde{x}_{it})=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

Next, use $\delta'(\tilde{x_{it}})=\delta_0\tilde{x}_{it}$ and $\delta(\tilde{x}_{it})=\frac{\delta_0}{2}\tilde{x}_{it}^2$ and N_t is large :

$$\Rightarrow\frac{(1-\alpha)\tilde{x}_{it}^2\delta_0}{r+\frac{\delta_0}{2}\tilde{x}_{it}^2-g_\pi}=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow(1-\alpha)\tilde{x}_{it}^2\delta_0=\left(r+\frac{\delta_0}{2}\tilde{x}_{it}^2-g_\pi\right)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow\tilde{x}_{it}^2\delta_0\left[(1-\alpha)-\frac{(\epsilon-1)}{2\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}\right]=(r-g_\pi)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow\tilde{x}_{it}=\left(\frac{(r-g_\pi)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}}{\delta_0(1-\alpha)-\delta_0\frac{(\epsilon-1)}{2\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}}\right)^{\frac{1}{2}}$$

When N_t is large, $\lim_{N_t\to\infty}f(N_t)=\frac{(1-\frac{1}{\epsilon})}{(1-\alpha)}$.

$$\Rightarrow\tilde{x}_{it}=\left(\frac{(r-g_\pi)\,2\,(\epsilon-1)\,(1-\alpha)\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}{2\delta_0(1-\alpha)-\delta_0(\epsilon-1)(1-\alpha)\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \tilde{x}_{it} = \left(\frac{(r - g_\pi)\, 2\,(\epsilon - 1)\, \eta \frac{(\sigma-1)}{\epsilon - \sigma\eta(\epsilon-1)}}{2\delta_0 - (\epsilon - 1)\delta_0 \eta \frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}} \right)^{\frac{1}{2}}$$

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\,(r - g_\pi) \left[(\epsilon - 1)\, \eta \frac{(\sigma-1)}{\epsilon - \sigma\eta(\epsilon-1)}\right]}{\delta_0 (2 - \left[(\epsilon - 1)\eta \frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}\right])} \right)^{\frac{1}{2}}$$

Define

$$\Gamma = (\epsilon - 1)\, \eta \frac{(\sigma - 1)}{\epsilon - \sigma\eta\,(\epsilon - 1)} = \frac{(\sigma - 1)}{\frac{\epsilon}{(\epsilon-1)\eta} - \sigma} = \frac{\eta(\sigma - 1)}{\frac{\epsilon}{\epsilon - 1} - \sigma\eta}. \tag{A.26}$$

Then:

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\,(r - g_\pi)\,\Gamma}{\delta_0 (2 - \Gamma)} \right)^{\frac{1}{2}}$$

Since $\rho = r - g_\pi$ then:

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\rho\Gamma}{\delta_0 (2 - \Gamma)} \right)^{\frac{1}{2}}. \tag{A.27}$$

B.6.2 Data used by own firm

From A.1 above:

$$x_f = 1$$

B.6.3 Firm Size

Use A.24:

$$L_{it} = \left(\frac{L_{pt}}{N_t} \right) = \frac{\chi \left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}} \right) (\sigma - 1)}{1 - \sigma(1 - \alpha)\eta f(N_t)} \quad (\frac{g_L}{g_L + \delta(\tilde{x}_{it})})$$

When N_t is large:

$$\begin{aligned}
L_{it} &= \frac{\chi\left(r+\delta(\tilde{x}_{it})-\frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)}{1-\sigma\eta(1-\frac{1}{\epsilon})} \quad (\frac{g_L}{g_L+\delta(\tilde{x}_{it})}) \\
&= \frac{\chi\left(\rho+\delta(\tilde{x}_{it})\right)(\sigma-1)}{1-\sigma\eta(1-\frac{1}{\epsilon})} \quad (\frac{g_L}{g_L+\delta(\tilde{x}_{it})}) \\
&= \chi g_L \frac{\rho+\delta(\tilde{x}_{it})}{g_L+\delta(\tilde{x}_{it})} \cdot \frac{\sigma-1}{1-\sigma\eta(\frac{\epsilon-1}{\epsilon})}
\end{aligned}$$

And define

$$\nu_f = \chi g_L \frac{\rho+\delta(\tilde{x}_{it})}{g_L+\delta(\tilde{x}_{it})} \cdot \frac{\sigma-1}{1-\sigma\eta(\frac{\epsilon-1}{\epsilon})}$$

These are equations (72) and (75).

B.6.4 Number of Varieties

From A.25, $N_t = L_t\psi(N_t)$ with $\psi(N_t) = \frac{1}{\nu(N_t)+\chi g_L}$ and $\nu(N_t) = \frac{\chi(r+\delta(\tilde{x}_{it})-g_\pi)(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}(\frac{g_L}{g_L+\delta(\tilde{x}_{it})})$. Now, from above, we know that when N_t is large $\nu = \chi g_L \frac{\rho+\delta(\tilde{x}_{it})}{g_L+\delta(\tilde{x}_{it})} \cdot \frac{\sigma-1}{1-\sigma\eta(\frac{\epsilon-1}{\epsilon})}$. Thus:

$$N_t = \frac{1}{\nu_f+\chi g_L} L_t = \psi_f L_t$$

B.6.5 Aggregate output

From A.14, $Y_t = N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. But $(\frac{L_{pt}}{N_t})^f = \nu_f$ Therefore:

$$Y_t^f = (N_t)^{\frac{\sigma}{\sigma-1}} (N_t)^{\frac{\eta}{1-\eta}} (\frac{\alpha}{N_t}+(1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} (\nu_f)^{\frac{1}{1-\eta}}$$

When N_t is large:

$$\begin{aligned}
Y_t^f &= (N_t)^{\frac{\sigma}{\sigma-1}} (N_t)^{\frac{\eta}{1-\eta}} ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} (\nu_f)^{\frac{1}{1-\eta}} \\
&= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (N_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}
\end{aligned}$$

and from equation (79), $N_t = \psi_f L_t$, thus:

$$Y_t^f = (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$$

B.6.6 Consumption per capita

$$c_t^f = \frac{Y_t^f}{L_t} \propto L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$$

This is equation (83) in the paper.

B.6.7 Consumption per capita growth

Using equation (83):

$$g_c^f = (\frac{1}{\sigma - 1} + \frac{\eta}{1-\eta})g_L$$

This is equation (85) in the paper.

B.6.8 Firm production

Combining equation (80) $Y_t^f = (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$ and A.8:

$$\begin{aligned} Y_{it}^f &= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} N_t^{-\frac{\sigma}{\sigma-1}} \\ &= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} (\psi_f L_t)^{-\frac{\sigma}{\sigma-1}} \\ &= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}} \end{aligned}$$

and this is equation 91 in the paper.

B.6.9 Data Production

Combining equation 91 and A.12:

$$\begin{aligned} D_{it}^f &= Y_{it}(\alpha + (1-\alpha)N_t\tilde{x}_{it}) \\ &= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}} N_t(\frac{\alpha}{N_t} + (1-\alpha)\tilde{x}_{it}) \\ &= (\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}+1} (1-\alpha)\tilde{x}_{it} \\ &= (\nu_f(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}} \end{aligned}$$

and this is Equation 87 in the paper.

B.6.10 Aggregate data production

By definition $D_t = N_t D_{it}$, hence:

$$\begin{aligned} D_t^f &= \psi_f L_t(\nu_f(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}} \\ &= (\nu_f(1-\alpha)\tilde{x}_{it})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{1-\eta}+1} \end{aligned}$$

and this is exactly equation 89 in the paper.

B.6.11 Labor share

From A.18:

$$\left(\frac{w_t L_{pt}}{Y_t}\right)^f = \frac{(1-\frac{1}{\sigma})}{1-\eta\frac{\epsilon}{\epsilon-1}} = \frac{\sigma}{\sigma\left(1-\eta\frac{\epsilon}{\epsilon-1}\right)}$$

which is equation (93).

B.6.12 Profit share

From A.21, $\pi_t = (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)$ and when N_t is large:

$$\begin{aligned}
\pi_t &= (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(\frac{(1-(1-\alpha)\eta f(N_t))\,\sigma - (\sigma-1)}{(1-(1-\alpha)\eta f(N_t))\,\sigma}\right) \\
&= ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+\frac{\eta}{1-\eta}} \left(\frac{1-(1-\alpha)\eta f(N_t)\sigma}{(1-(1-\alpha)\eta f(N_t))\,\sigma}\right) \\
&= ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+\frac{\eta}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{(1-\eta(\frac{\epsilon-1}{\epsilon}))\,\sigma}\right)
\end{aligned}$$

Hence:

$$
\begin{aligned}
\left(\frac{\pi_t N_t}{Y_t}\right)^f &= \frac{((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1+\frac{\eta}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}\right)}{(\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}} \\
&= \frac{L_{pt}^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}\right)}{(\nu_f)^{\frac{1}{1-\eta}}} \\
&= \frac{(\nu_f N_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}\right)}{(\nu_f)^{\frac{1}{1-\eta}}} \\
&= (N_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}\right) \\
&= (\psi_f L_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \frac{1-\eta\sigma\left(\frac{\epsilon-1}{\epsilon}\right)}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma} \\
&= \frac{1-\eta\sigma\left(\frac{\epsilon-1}{\epsilon}\right)}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}
\end{aligned}
$$

which is equation (94).

B.6.13 Data Share

Use $p_{at} = \frac{\alpha}{1-\alpha} p_{bt}$ to value the data the firm owns, using the perfect substitutes argument. Then,

$$
\begin{aligned}
p_{at} Y_{it} + p_{bt} D_{bt} &= p_{bt}\left(\frac{\alpha}{1-\alpha} Y_{it} + D_{bt}\right) \\
&= \frac{p_{bt}}{1-\alpha}(\alpha Y_{it} + (1-\alpha) D_{bt}) \\
&= \frac{p_{bt}}{1-\alpha} D_{it} \\
&= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t] Y_{it} \\
&= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t] Y_t N_t^{\frac{-\sigma}{\sigma-1}}
\end{aligned}
$$

Now recall from (A.14) that $Y_t = N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. Substituting this into the last equation above gives

$$\begin{aligned} p_{at}Y_{it} + p_{bt}D_{bt} &= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t] N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-\sigma}{\sigma-1}} \\ &= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{1}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-1}{1-\eta}} \end{aligned}$$

Now recall from (A.15) that $\frac{p_{bt}}{1-\alpha} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\eta(\sigma-1)}-1} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}}$ and substitute this in to get

$$p_{at}Y_{it} + p_{bt}D_{bt} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\eta(\sigma-1)}-1} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} [\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{1}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-1}{1-\eta}}$$

Again use equation (A.14) for Y_t to substitute for $Y_t^{-\frac{1}{\eta}}$ and simplify the exponents to get

$$p_{at}Y_{it} + p_{bt}D_{bt} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right)\frac{Y_t}{N_t}$$

which gives the data share of GDP as

$$\frac{N_t(p_{at}Y_{it} + p_{bt}D_{bt})}{Y_t} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right)$$

Finally, using the result for the labor share in equation (93), we have our result:

$$\frac{N_t(p_{at}Y_{it} + p_{bt}D_{bt})}{Y_t} = \frac{\eta}{1-\eta\frac{\epsilon-1}{\epsilon}} \cdot \frac{\sigma-1}{\sigma}$$

which is equation (101).

B.6.14 Price of a variety

From household problem, $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$, then:

$$\begin{aligned} p_{it}^f &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \\ &= \left(\frac{Y_t}{Y_t N_t^{-\frac{\sigma}{\sigma-1}}}\right)^{\frac{1}{\sigma}} \\ &= N_t^{\frac{1}{\sigma-1}} \end{aligned}$$

and from equation (79), $N_t = \psi_f L_t$, thus:

$$p_{it}^f = (\psi_f L_t)^{\frac{1}{\sigma-1}}$$

which is equation (98).

C Competitive Equilibrium When Consumers Own Data

C.1 Household Problem

The household problem is

$$\begin{aligned} U_0 = & \max_{\{c_{it}, x_{it}, \tilde{x}_{it}\}} \int_0^\infty e^{-(\tilde{\rho})} L_0 u(c_t, x_{it}, \tilde{x}_{it}) dt \\ s.t.\ c_t = & \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\ \dot{a}_t = & (r_t - g_L) a_t + w_t - \int_0^{N_t} p_{it} c_{it} di + \int_0^{N_t} x_{it} p_{st}^a c_{it} di + \int_0^{N_t} \tilde{x}_{it} p_{st}^b c_{it} di \\ = & (r_t - g_L) a_t + w_t - \int_0^{N_t} q_{it} c_{it} di \end{aligned}$$

The Hamiltonian of the problem is

$$H(c_{it}, x_{it}, \tilde{x}_{it}, a_t, \mu_t) = u\left(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it}\right) + \mu_t \left[(r_t - g_L) a_t + w_t - \int_0^{N_t} q_{it} c_{it} di\right]$$

The FOCs are:

$$\begin{cases} \frac{\partial H}{\partial c_{it}} = 0 \\ \frac{\partial H}{\partial x_{it}} = 0 \\ \frac{\partial H}{\partial \tilde{x}_{it}} = 0 \\ \frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t \end{cases}$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma-1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t q_{it} = 0$$

$$\Rightarrow \; c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t q_{it} = 0$$

$$\Rightarrow \; c_t^{\sigma-1} c_{it} = (\mu_t q_{it})^{-\sigma}$$

$$\Rightarrow \; \left(c_t^{\sigma-1} c_{it}\right)^{\frac{\sigma-1}{\sigma}} = (\mu_t q_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (q_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{\sigma-1}{\sigma} + \frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow \; \mu_t = \frac{1}{c_t \left(\int_0^{N_t} q_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}}$$

Next, define $P_t = \left(\int_0^{N_t} q_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}$. Thus:

$$\Rightarrow \mu_t = \frac{1}{c_t P_t} \tag{A.28}$$

Next, plug the expression for μ_t in the FOC:

$$c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \frac{1}{c_t P_t} q_{it} = 0$$

$$\Rightarrow c_{it} = c_t \left(\frac{q_{it}}{P_t}\right)^{-\sigma}$$

Using the normalization $P_t = 1$ yields :

$$c_{it} = c_t \left(q_{it}\right)^{-\sigma} \tag{A.29}$$

and therefore we get equation (52) $q_{it} = \left(\frac{c_t}{c_{it}}\right)^{\frac{1}{\sigma}}$.

Next, compute the FOC $\frac{\partial H}{\partial x_{it}} = 0$:

$$\frac{\partial H}{\partial x_{it}} = 0$$
$$\frac{\kappa}{N_t} x_{it} = \mu_t p_{st}^a c_{it} N_t$$
$$\frac{\kappa}{N_t^2} x_{it} = \mu_t p_{st}^a c_{it}$$

Next, compute the FOC $\frac{\partial H}{\partial \tilde{x}_{it}} = 0$:

$$\frac{\partial H}{\partial \tilde{x}_{it}} = 0$$
$$\frac{\tilde{\kappa}}{N_t} \tilde{x}_{it} = \mu_t p_{st}^b c_{it}$$

Finally, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t(r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$

$$\Rightarrow (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} + g_c$$

Next, use A.28 and A.29 in the FOC of x_{it}:

$$\frac{\kappa}{N_t^2} x_{it} = p_{st}^a \left(q_{it}\right)^{-\sigma}$$

Thus:

$$x_{it} = \frac{N_t^2 p_{st}^a}{\kappa} \left(q_{it}\right)^{-\sigma} \tag{A.30}$$

Similarly, use A.28 and A.29 in the FOC of $\tilde{x}_{it}$:

$$\frac{\tilde{\kappa}}{N_t} \tilde{x}_{it} = p_{st}^b$$

$$\tilde{x}_{it} = \frac{N_t}{\tilde{\kappa}} \frac{1}{c_t} p_{st}^b c_t \left(q_{it}\right)^{-\sigma}$$

$$\Rightarrow \tilde{x}_{it} = \frac{N_t}{\tilde{\kappa}} p_{st}^b \left(q_{it}\right)^{-\sigma} \tag{A.31}$$

Now, we know that $q_{it} = \left(\frac{c_t}{c_{it}}\right)^{\frac{1}{\sigma}} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$ and that $q_{it} = p_{it} - x_{it} p_{st}^a - \tilde{x}_{it} p_{st}^b$ then :

$$p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it} p_{st}^a + \tilde{x}_{it} p_{st}^b \tag{A.32}$$

which is equation (53).

C.2 Firm Problem

The firm problem is given by:

$$r_tV_{it} = \max_{\{L_{it},D_{ait},D_{bit}\}} \left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st} \right] Y_{it} - w_tL_{it} - p_{bt}D_{bit} - p_{at}D_{ait} - \delta(\tilde{x_{it}})V_{it} + \dot{V}_{it}$$

$$s.t.\ Y_{it} = D^{\eta}_{it}L_{it}$$

$$D_{it} = \alpha D_{ait} + (1-\alpha)D_{bit}$$

$$D_{ait} \geq 0$$

$$D_{bit} \geq 0$$

Start with the FOC w/r to L_{it}:

$$\frac{\partial \mathbb{L}}{\partial L_{it}} = 0$$

$$\Leftrightarrow\ (-\frac{1}{\sigma})Y_t^{\frac{1}{\sigma}}Y_{it}^{\frac{-1}{\sigma}-1}\frac{\partial Y_{it}}{\partial L_{it}}Y_{it} + \frac{\partial Y_{it}}{\partial L_{it}}\left(\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right) - w_t = 0$$

$$\frac{\partial Y_{it}}{\partial L_{it}}\left[(-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = w_t$$

But $\frac{\partial Y_{it}}{\partial L_{it}} = D^{\eta}_{it} = \frac{Y_{it}}{L_{it}}$ and therefore the FOC is:

$$\frac{Y_{it}}{L_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = w_t$$

$$\Rightarrow \frac{Y_{it}}{L_{it}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = w_t \qquad \text{(A.33)}$$

Next, the FOC w/r to D_{ait}:

$$\frac{\partial \mathbb{L}}{\partial D_{ait}} = 0$$

$$\Leftrightarrow\ (-\frac{1}{\sigma})Y_t^{\frac{1}{\sigma}}Y_{it}^{\frac{-1}{\sigma}-1}\frac{\partial Y_{it}}{\partial D_{ait}}Y_{it} + \frac{\partial Y_{it}}{\partial D_{ait}}\left(\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right) - p_{at} = 0$$

$$\frac{\partial Y_{it}}{\partial D_{ait}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = p_{at}$$

Next, compute the derivative $\frac{\partial Y_{it}}{\partial D_{ait}}$:

$$\frac{\partial Y_{it}}{\partial D_{ait}} = \frac{\partial Y_{it}}{\partial D_{it}} \frac{\partial D_{it}}{\partial D_{ait}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial D_{ait}} = \eta \frac{Y_{it}}{D_{it}} \alpha$$

Thus the FOC w/r to D_{ait} is:

$$\eta\alpha \frac{Y_{it}}{D_{it}} \left[\frac{\sigma - 1}{\sigma} \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st} \right] = p_{at} \tag{A.34}$$

Similarly, the FOC w/r to D_{bit} is:

$$\eta(1-\alpha) \frac{Y_{it}}{D_{it}} \left[\frac{\sigma - 1}{\sigma} \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st} \right] = p_{bt} \tag{A.35}$$

We have computed the 3 FOCs. Now I solve the problem. First, divide A.34 by A.35 to get:

$$p_{at} = \frac{\alpha}{(1-\alpha)} p_{bt} \tag{A.36}$$

Second, divide A.33 by A.35:

$$\frac{Y_{it}}{L_{it}} \left[\eta(1-\alpha) \frac{Y_{it}}{D_{it}} \right]^{-1} = \frac{w_t}{p_{bt}}$$

$$\frac{D_{it}}{\eta(1-\alpha) L_{it}} = \frac{w_t}{p_{bt}}$$

$$\Rightarrow D_{it} = \frac{w_t}{p_{bt}} \eta(1-\alpha) L_{it}$$

but we knot that $Y_{it} = D^{\eta}_{it} L_{it}$, therefore $L_{it} = \frac{Y_{it}}{D^{\eta}_{it}}$. Substituting L_{it}:

$$D^{\eta+1}_{it} = \frac{w_t}{p_{bt}} \eta(1-\alpha) Y_{it}$$

$$D_{it} = \left[\frac{w_t}{p_{bt}} \eta(1-\alpha) Y_{it} \right]^{\frac{1}{\eta+1}}$$

but $q_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$ therefore $Y_{it} = Y_t q_{it}^{-\sigma}$. Substituting:

$$D_{it} = \left[\frac{w_t}{p_{bt}}\eta(1-\alpha)Y_t q_{it}^{-\sigma}\right]^{\frac{1}{\eta+1}} \tag{A.37}$$

On the other hand, using A.33 :

$$D_{it}^{\eta}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^{a} + \tilde{x}_{it}p_{st}^{b}\right] = w_t$$

$$\Rightarrow D_{it}^{\eta}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^{a} + \tilde{x}_{it}p_{st}^{b}\right] = w_t$$

$$\Rightarrow \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} = \frac{\sigma}{\sigma-1}\left(\frac{w_t}{D_{it}^{\eta}} - x_{it}p_{st}^{a} - \tilde{x}_{it}p_{st}^{b}\right)$$

$$\Rightarrow Y_{it} = Y_t\left[\frac{\sigma}{\sigma-1}\left(\frac{w_t}{D_{it}^{\eta}} - x_{it}p_{st}^{a} - \tilde{x}_{it}p_{st}^{b}\right)\right]^{-\sigma} \tag{A.38}$$

C.3 The 2 Data Intermediary Problems

The 2 data intermediary problems are given by:

$$\max_{p_{ait}, D_{cit}^{a}} \int_0^{N_t} p_{ait}D_{ait}di - \int_0^{N_t} p_{st}^{a}D_{cit}^{a}di$$
$$s.t. D_{ait} \le D_{cit}^{a}$$
$$p_{ait} \le p_{ait}^{*}$$

and

$$\max_{p_{bit}, D_{cit}^{b}} \int_0^{N_t} p_{bit}D_{bit}di - \int_0^{N_t} p_{st}^{b}D_{cit}^{b}di$$
$$s.t. D_{bit} \le B_t = \left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t} D_{cit}^{b\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}$$
$$p_{bit} \le p_{bit}^{*}$$

As in the allocation when firms own the data, these problems can be decomposed in a cost minimization problem and a zero profit condition.

C.3.1 The Cost Minimization Problem

The cost minimization problems are:

$$\min_{D^a_{cit}} \int_0^{N_t} p^a_{st} D^a_{cit} di$$
$$s.t. D_{ait} \leq D^a_{cit}$$

and

$$\min_{D^b_{cit}} \int_0^{N_t} p^b_{st} D^b_{cit} di$$
$$s.t. D_{bit} \leq B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D^{b\frac{\epsilon-1}{\epsilon}}_{cit} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

The solutions are given by:

$$\begin{cases} D_{ait} = D^a_{cit} \\ D_{bit} = B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D^{b\frac{\epsilon-1}{\epsilon}}_{cit} di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t D^b_{cit} \end{cases} \tag{A.39}$$

where D_{ait} and D_{bit} are given by the firm problem.

C.3.2 The Zero Profit Condition for Each Data Intermediary Problem

By symmetry, the profit for each data intermediary is:

$$\begin{cases} \pi_{ait} = N_t \left[p_{ait} D_{ait} - p^a_{st} D^a_{cit} \right] = N_t D_{ait} \left[p_{ait} - p^a_{st} \right] \\ \pi_{bit} = N_t \left[p_{bit} D_{bit} - p^b_{st} D^b_{cit} \right] = N_t D_{bit} \left[p_{bit} - \frac{p^b_{st}}{N_t} \right] \end{cases}$$

The zero profit condition yields:

$$\begin{cases} p_{ait} = p^a_{st} \\ p_{bit} = \frac{p^b_{st}}{N_t} \end{cases} \tag{A.40}$$

and these are exactly equation (63).

C.4 Market Clearing Condition for Data

Imposing the market clearing condition in the data market, we get:

$$\begin{aligned} D^a_{cit} &= x_{it} c_{it} L_t \\ D^b_{cit} &= \tilde{x}_{it} c_{it} L_t \end{aligned} \tag{A.41}$$

but from the data intermediary problem $D_{bit} = B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D_{cit}^{b \frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t D^b_{cit}$ and $D_{ait} = D^a_{cit}$ Thus the market clearing conditions yield

$$\begin{aligned} D_{ait} &= x_{it} c_{it} L_t \\ D_{bit} &= \tilde{x}_{it} c_{it} L_t N_t \end{aligned} \tag{A.42}$$

and therefore the supply of data is given by:

$$\begin{aligned} D_{it} &= \alpha D_{ait} + (1-\alpha) D_{bit} \\ &= \alpha x_{it} c_{it} L_t + (1-\alpha) \tilde{x}_{it} c_{it} L_t N_t \\ &= c_{it} L_t \left[\alpha x_{it} + (1-\alpha) \tilde{x}_{it} N_t \right] \\ &= Y_{it} \left[\alpha x_{it} + (1-\alpha) \tilde{x}_{it} N_t \right] \end{aligned} \tag{A.43}$$

Moreover, the zero profit condition means:

$$\pi_{ait} + \pi_{bit} = 0$$

$$\Rightarrow p_{at} D_{ait} + p_{bt} D_{bit} = p^a_{st} D^a_{cit} + D^b_{cit} p^b_{st}$$

Next, use A.39:

$$\Rightarrow p_{at} D^a_{cit} + p_{bt} N_t D^b_{cit} = p^a_{st} D^a_{cit} + D^b_{cit} p^b_{st}$$

Substitute using A.42:

$$\Rightarrow p_{at} x_{it} c_{it} L_t + p_{bt} N_t \tilde{x}_{it} c_{it} L_t = p^a_{st} x_{it} c_{it} L_t + \tilde{x}_{it} c_{it} L_t p^b_{st}$$

$$\Rightarrow p_{at}x_{it} + p_{bt}N_t\tilde{x}_{it} = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$$

Use A.36:

$$\Rightarrow \frac{\alpha}{(1-\alpha)}p_{bt}x_{it} + p_{bt}N_t\tilde{x}_{it} = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$$

And we finally get:

$$\Rightarrow \frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it}\right] = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st} \tag{A.44}$$

C.5 Summary of Equations and Solution

C.5.1 Summary

Let's summarize all the equations we have so far, from the FOCs of the consumer and firm problems:

$$\begin{cases} N_t^{-\frac{\sigma}{\sigma-1}} = \frac{Y_{it}}{Y_t} = \frac{c_{it}}{c_t} = (q_{it})^{-\sigma} \\ x_{it} = \frac{N_t^2 p_{st}^a}{\kappa} (q_{it})^{-\sigma} \\ \tilde{x}_{it} = \frac{N_t}{\tilde{\kappa}} p_{st}^b (q_{it})^{-\sigma} \\ \frac{Y_{it}}{L_{it}} \left[\frac{\sigma-1}{\sigma} \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + x_{it} p_{st}^a + \tilde{x}_{it} p_{st}^b \right] = w_t \\ \eta\alpha \frac{Y_{it}}{D_{it}} \left[\frac{\sigma-1}{\sigma} \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + x_{it} p_{st}^a + \tilde{x}_{it} p_{st}^b \right] = p_{at} \\ p_{at} = \frac{\alpha}{(1-\alpha)} p_{bt} \\ D_{it} = \left[\frac{w_t}{p_{bt}} \eta (1-\alpha) Y_t q_{it}^{-\sigma} \right]^{\frac{1}{\eta+1}} \\ Y_{it} = Y_t \left[\frac{\sigma}{\sigma-1} \left(\frac{w_t}{D_{it}^{\eta}} - x_{it} p_{st}^a - \tilde{x}_{it} p_{st}^b \right) \right]^{-\sigma} \\ D_{ait} = D_{cit}^a \\ D_{bit} = N_t D_{cit}^b \\ p_{at} = p_{st}^a \\ p_{bt} = \frac{p_{st}^b}{N_t} \\ D_{ait} = x_{it} c_{it} L_t \\ D_{bit} = \tilde{x}_{it} c_{it} L_t N_t \\ \frac{p_{bt}}{(1-\alpha)} \left[\alpha x_{it} + (1-\alpha) N_t \tilde{x}_{it} \right] = p_{st}^a x_{it} + \tilde{x}_{it} p_{st}^b \\ D_{it} = Y_{it} \left[\alpha x_{it} + (1-\alpha) \tilde{x}_{it} N_t \right] \end{cases} \tag{A.45}$$

C.5.2 Solution for x_{it} and $\tilde{x}_{it}$

First, use $(q_{it})^{-\sigma} = N_t^{-\frac{\sigma}{\sigma-1}}$ in A.31 and A.30 to get:

$$x_{it} = \frac{p_{st}^a}{\kappa} N_t^{2-\frac{\sigma}{\sigma-1}} \tag{A.46}$$

$$\tilde{x}_{it} = \frac{p_{st}^b}{\tilde{\kappa}} N_t^{1-\frac{\sigma}{\sigma-1}} \tag{A.47}$$

Next, using $N_t^{-\frac{\sigma}{\sigma-1}} = \frac{Y_{it}}{Y_t}$ and A.43 in A.34:

$$\eta\alpha\frac{Y_{it}}{D_{it}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = p_{at}$$

$$\Rightarrow \eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at} \tag{A.48}$$

Now, use A.46 and A.47 to substitute p^a_{st} and p^b_{st}:

$$\eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x^2_{it}\kappa N_t^{-2+\frac{\sigma}{\sigma-1}} + \tilde{x}^2_{it}\tilde{\kappa}N_t^{-1+\frac{\sigma}{\sigma-1}}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$\Longrightarrow \eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x^2_{it}\kappa N_t^{\frac{2-\sigma}{\sigma-1}} + \tilde{x}^2_{it}\tilde{\kappa}N_t^{\frac{1}{\sigma-1}}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$\Longrightarrow \eta\alpha\frac{N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$\Longrightarrow \eta\alpha\frac{N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

Use $p_{at} = \frac{\alpha}{(1-\alpha)}p_{bt}$ and get:

$$\Longrightarrow \eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right] = \frac{p_{bt}}{1-\alpha}\left(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right) \tag{A.49}$$

Now, recall from A.44: $\frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it}\right] = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$. With this, substitute the RHS in A.49:

$$\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right] = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$$

Again, use A.46 and A.47 to substitute p^a_{st} and p^b_{st}:

$$\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right] = x^2_{it}\kappa N_t^{\frac{2-\sigma}{\sigma-1}} + \tilde{x}^2_{it}\tilde{\kappa}N_t^{\frac{1}{\sigma-1}}$$

$$\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma}+x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right]=N_t^{\frac{1}{\sigma-1}}\left[x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right]$$

$$\eta\frac{\sigma-1}{\sigma}+\eta x_{it}^2\kappa N_t^{-1}+\eta\tilde{x}_{it}^2\tilde{\kappa}=x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}$$

$$\eta\frac{\sigma-1}{\sigma}+(\eta-1)x_{it}^2\kappa N_t^{-1}+(\eta-1)\tilde{x}_{it}^2\tilde{\kappa}=0$$

Finally:

$$\Rightarrow x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}=\frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma} \tag{A.50}$$

On other hand, divide A.46 by A.47:

$$\frac{x_{it}}{\tilde{x}_{it}}=\frac{p_{st}^a N_t\tilde{\kappa}}{p_{st}^b\kappa}$$

Use the zero profit condition for the data intermediary problems:

$$\Rightarrow\frac{x_{it}}{\tilde{x}_{it}}=\frac{p_{at}N_t\tilde{\kappa}}{p_{bt}N_t\kappa}$$

$$\Rightarrow\frac{x_{it}}{\tilde{x}_{it}}=\frac{p_{at}\tilde{\kappa}}{p_{bt}\kappa}$$

Use A.36:

$$\Rightarrow\frac{x_{it}}{\tilde{x}_{it}}=\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa} \tag{A.51}$$

Hence, use A.51 in A.50:

$$\tilde{x}_{it}^2\kappa N_t^{-1}\left(\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}\right)^2+\tilde{x}_{it}^2\tilde{\kappa}=\frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}$$

When N_t is large, $\underset{N_t\to\infty}{lim}N_t^{-1}\left(\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}\right)^2=0$, therefore:

$$\tilde{x}^c = \left[\frac{1}{\tilde{\kappa}} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma}\right]^{\frac{1}{2}} \tag{A.52}$$

which is equation (65). Moreover, the expression for x^c is:

$$x^c = \frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}\tilde{x}^c$$

which is equation (69).

C.5.3 The prices p_{at} and p_{bt}

From A.49:

$$p_{bt} = (1-\alpha)\eta N_t^{\frac{1}{\sigma-1}} \frac{\left[\frac{\sigma-1}{\sigma} + x_{it}^2\kappa N_t^{-1} + \tilde{x}_{it}^2\tilde{\kappa}\right]}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)}$$

In the numerator, when N_t is large:

$$\begin{aligned}
p_{bt} &= (1-\alpha)\eta N_t^{\frac{1}{\sigma-1}} \frac{\left[\frac{\sigma-1}{\sigma} + \tilde{x}_{it}^2\tilde{\kappa}\right]}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)} \\
&= (1-\alpha)\eta N_t^{\frac{1}{\sigma-1}} \frac{\left[\frac{\sigma-1}{\sigma} + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right]}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)} \\
&= (1-\alpha)\eta N_t^{\frac{1}{\sigma-1}} \frac{1}{1-\eta}\frac{\sigma-1}{\sigma}\frac{1}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)} \\
&= (1-\alpha)N_t^{\frac{1}{\sigma-1}} \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\frac{1}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)}
\end{aligned} \tag{A.53}$$

C.5.4 Wage w_t

Divide A.33 by A.34:

$$w_t = p_{at}\frac{D_{it}}{L_{it}\eta\alpha}$$

Substitute D_{it} and L_{it}:

$$w_t = p_{at}\frac{N_t Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right]}{L_{pt}\eta\alpha}$$

But $N_t^{-\frac{\sigma}{\sigma-1}} = \frac{Y_{it}}{Y_t}$, therefore:

$$w_t = p_{at}\frac{N_t^{1-\frac{\sigma}{\sigma-1}}Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right]}{L_{pt}\eta\alpha}$$

On the other hand, $p_{at} = \frac{\alpha}{(1-\alpha)}p_{bt} = \alpha\eta N_t^{\frac{1}{\sigma-1}}\frac{\left[\frac{\sigma-1}{\sigma}+x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right]}{\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t}$. Substitute:

$$w_t = \alpha\eta N_t^{\frac{1}{\sigma-1}}\frac{\left[\frac{\sigma-1}{\sigma} + x_{it}^2\kappa N_t^{-1} + \tilde{x}_{it}^2\tilde{\kappa}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t}\frac{N_t^{1-\frac{\sigma}{\sigma-1}}Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right]}{L_{pt}\eta\alpha}$$

$$w_t = \left[\frac{\sigma-1}{\sigma} + x_{it}^2\kappa N_t^{-1} + \tilde{x}_{it}^2\tilde{\kappa}\right]\frac{Y_t}{L_{pt}}$$

When N_t is large:

$$w_t = \left[\frac{\sigma-1}{\sigma} + \tilde{x}_{it}^2\tilde{\kappa}\right]\frac{Y_t}{L_{pt}}$$

Finally, substitute $\tilde{x}_{it}$:

$$w_t = \left[\frac{\sigma-1}{\sigma} + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right]\frac{Y_t}{L_{pt}}$$

$$\Rightarrow w_t = \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{L_{pt}} \tag{A.54}$$

C.5.5 Value of the Firm and Profits

By definition, similarly to the case when firms own the data, the value of the firm is given by:

$$V_t = \frac{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right]Y_{it} - w_tL_{it} - p_{bt}D_{bit} - p_{at}D_{ait}}{r + \delta(\tilde{x_{it}}) - g_V}$$

Let's compute each component separately:

$$V_t = \frac{\overbrace{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right]Y_{it}}^{C} - \overbrace{w_tL_{it}}^{B}\overbrace{- p_{bt}D_{bit} - p_{at}D_{ait}}^{A}}{r + \delta(\tilde{x_{it}}) - g_V}$$

First, let's compute A:

$$\begin{aligned}
A &= p_{bt}D_{bit} + p_{at}D_{ait} \\
&= p_{bt}\left[D_{bit} + \frac{\alpha}{1-\alpha}D_{ait}\right] \\
&= \frac{p_{bt}}{(1-\alpha)}\left[(1-\alpha)D_{bit} + \alpha D_{ait}\right] \\
&= \frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it}c_{it}L_t + (1-\alpha)\tilde{x}_{it}c_{it}L_tN_t\right] \\
&= \frac{p_{bt}}{(1-\alpha)}Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right] \\
&= \frac{p_{bt}}{(1-\alpha)}N_t^{-\frac{\sigma}{\sigma-1}}Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right] \\
&= (1-\alpha)N_t^{\frac{1}{\sigma-1}}\frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\frac{1}{(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t)}\frac{1}{(1-\alpha)}N_t^{-\frac{\sigma}{\sigma-1}}Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right] \\
&= \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\frac{Y_t}{N_t}
\end{aligned}$$

Note, this term is equation (100), which gives the data share of the economy.

Next, let's compute B:

$$\begin{aligned}
B &= w_tL_{it} \\
&= \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{L_{pt}}L_{it} \\
&= \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{L_{pt}}\frac{L_{pt}}{N_t} \\
&= \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{N_t}
\end{aligned}$$

Finally, compute C:

$$\begin{aligned}
C &= \left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] Y_{it} \\
&= \left[N_t^{\frac{1}{\sigma-1}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] Y_{it} \\
&= \left[N_t^{\frac{1}{\sigma-1}} + x_{it}^2\kappa N_t^{-2+\frac{\sigma}{\sigma-1}} + \tilde{x}_{it}^2\tilde{\kappa}\right] Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}}\left[1 + x_{it}^2\kappa N_t^{-1} + \tilde{x}_{it}^2\tilde{\kappa}\right] Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}}\left[1 + \tilde{x}_{it}^2\tilde{\kappa}\right] Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}}\left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}}\left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\
&= \frac{Y_t}{N_t}\left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right]
\end{aligned}$$

Using (A), (B),and (C) we can compute the value of the firm:

$$\begin{aligned}
V_{it} &= \frac{\overbrace{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] Y_{it}}^{C} - \overbrace{w_t L_{it}}^{B} \overbrace{- p_{bt}D_{bit} - p_{at}D_{ait}}^{A}}{r + \delta(\tilde{x_{it}}) - g_V} \\
&= \frac{1}{r + \delta(\tilde{x_{it}}) - g_V}\left[\frac{Y_t}{N_t}\left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] - \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{N_t} - \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\frac{Y_t}{N_t}\right] \\
&= \frac{Y_t}{N_t}\frac{1}{(r + \delta(\tilde{x_{it}}) - g_V)}\frac{(1-\sigma\eta)}{\sigma(1-\eta)}
\end{aligned}$$

C.6 Solution of the Competitive Equilibrium

C.6.1 Firm Size

Next, impose the free entry condition:

$$\chi w_t = V_{it}(1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

Substitute V_{it} and w_t:

$$\chi\frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{L_{pt}}=\frac{Y_t}{N_t}\frac{1}{(r+\delta(\tilde{x_{it}})-g_V)}\frac{(1-\sigma\eta)}{\sigma(1-\eta)}(1+\frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow\frac{L_{pt}}{N_t}=\chi\frac{(\sigma-1)\,(r+\delta(\tilde{x_{it}})-g_V)}{(1-\sigma\eta)\,(1+\frac{\delta(\tilde{x}_{it})}{g_L})}$$

Use $\rho=r-g_V$ on BGP:

$$\Rightarrow\frac{L_{pt}}{N_t}=\chi\frac{(\sigma-1)\,(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)\,(1+\frac{\delta(\tilde{x}_{it})}{g_L})}$$

$$\Rightarrow\frac{L_{pt}}{N_t}=\chi\frac{(\sigma-1)\,(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)\,(1+\frac{\delta(\tilde{x}_{it})}{g_L})}$$

$$\Rightarrow\frac{L_{pt}}{N_t}=\chi g_L\frac{(\sigma-1)\,(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)\,(g_L+\delta(\tilde{x}_{it}))} \tag{A.55}$$

Define $\nu_c=\frac{L_{pt}}{N_t}$. Finally:

$$\Rightarrow v_c=\chi g_L\frac{(\sigma-1)\,(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)\,(g_L+\delta(\tilde{x}_{it}))}$$

which is equation (74) in the paper.

C.6.2 Number of Varieties

From the evolution of the number of varieties, we have $\dot{N}_t=\frac{1}{\chi}(L_t-L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t}=\frac{1}{\chi}(\frac{L_t}{N_t}-\frac{L_{pt}}{N_t})$$

In BGP: $\frac{\dot{N}_t}{N_t}=g_N$ and since $g_N=g_L$, then:

$$\Rightarrow g_L=\frac{1}{\chi}(\frac{L_t}{N_t}-\frac{L_{pt}}{N_t})$$

$$\Rightarrow\frac{L_t}{N_t}=\frac{L_{pt}}{N_t}+\chi g_L$$

Next, using $\frac{\dot{V}_{it}}{V_{it}}=g_\pi$ and substituting $\frac{L_{pt}}{N_t}$ using A.55:

$$\Rightarrow \frac{L_t}{N_t} = \chi g_L \frac{(\sigma - 1)(\varrho + \delta(\tilde{x_{it}}))}{(1 - \sigma\eta)(g_L + \delta(\tilde{x}_{it}))} + \chi g_L$$

$$\Rightarrow N_t = \frac{L_t}{\chi g_L \frac{(\sigma-1)(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)(g_L+\delta(\tilde{x}_{it}))} + \chi g_L}$$

Define $\nu_c = \chi g_L \frac{(\sigma-1)(\varrho+\delta(\tilde{x_{it}}))}{(1-\sigma\eta)(g_L+\delta(\tilde{x}_{it}))}$, so that:

$$N_t = \frac{L_t}{\nu_c + \chi g_L}$$

Moreover, define $\psi_c = \frac{1}{\nu_c + \chi g_L}$ so that:

$$N_t = \psi_c L_t$$

which is equation (79) for the allocation when consumers own the data.

C.6.3 Aggregate Output

First, substitute $D_{it} = Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right]$ in $Y_{it} = D_{it}^{\eta} L_{it}$.

$$\begin{aligned} Y_{it} &= \left(Y_{it}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})\right)^{\eta} L_{it} \\ &= Y_{it}^{\eta}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\eta} \left(\frac{L_{pt}}{N_t}\right) \end{aligned}$$

$$\Rightarrow Y_{it} = (\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \tag{A.56}$$

On the other hand, aggregate output is $Y_t = Y_{it} N_t^{\frac{\sigma}{\sigma-1}}$. Hence:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}} (\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \end{aligned}$$

When N_t is large:

$$\begin{aligned} Y_t^c &= (N_t)^{\frac{\sigma}{\sigma-1}} (N_t)^{\frac{\eta}{1-\eta}} ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} (\nu_c)^{\frac{1}{1-\eta}} \\ &= (\nu_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (N_t)^{\frac{\sigma}{\sigma-1} + \frac{\eta}{1-\eta}} \end{aligned}$$

and from equation (79), $N_t = \psi_c L_t$, thus:

$$Y_t^c = (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$$

C.6.4 Firm production

$$\begin{aligned} Y_{it}^c &= Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\ &(\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} N_t^{-\frac{\sigma}{\sigma-1}} \\ &= (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} (\psi_f L_t)^{-\frac{\sigma}{\sigma-1}} \\ &= (v_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\eta}{1-\eta}} \end{aligned}$$

which is equation (91) for allocation c.

C.6.5 Consumption per capita and growth

Consumption per capita is defined as $c_t^c = \frac{Y_t^c}{L_t}$, and using the expression for aggregate output, then:

$$c_t^c = \frac{Y_t^c}{L_t} \propto L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$$

which is equation (83) in the paper for the allocation c. Using this, the growth is:

$$g_c^c = (\frac{1}{\sigma-1} + \frac{\eta}{1-\eta})g_L$$

which is equation (85) in the paper for the allocation c.

C.6.6 Data used by each firm and aggregate data

$$\begin{aligned} D_{it}^c &= Y_{it}(\alpha x_{it} + (1-\alpha)N_t \tilde{x}_{it}) \\ &= (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\eta}{1-\eta}} N_t(\frac{\alpha}{N_t} x_{it} + (1-\alpha)\tilde{x}_{it}) \\ &= (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\eta}{1-\eta}+1} (1-\alpha)\tilde{x}_{it} \\ &= (\nu_c(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}} \end{aligned}$$

By definition $D_t = N_t D_{it}$, hence:

$$\begin{aligned} D_t^c &= \psi_c L_t (\nu_c(1-\alpha)\tilde{x}_{it}\psi_c L_t)^{\frac{1}{1-\eta}} \\ &= (\nu_c(1-\alpha)\tilde{x}_{it})^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{1}{1-\eta}+1} \end{aligned}$$

and this is equation (89).

C.6.7 Labor Share

From A.54:

$$\left(\frac{w_t L_{pt}}{Y_t}\right)^c = \frac{\sigma - 1}{\sigma(1-\eta)} \frac{Y_t}{L_{pt}} \frac{L_{pt}}{Y_t} = \frac{\sigma - 1}{\sigma(1-\eta)}$$

and this is equation (93) in the paper.

C.6.8 Profit share

Using the results from C.5.5, the profit is defined as $\pi_t = C - A - B$, where C, B and A are defined in C.5.5. Then:

$$\begin{aligned}\left(\frac{N_t \pi_t}{Y_t}\right) &= \frac{N_t}{Y_t}\left(\frac{Y_t}{N_t}\left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] - \frac{\sigma-1}{\sigma(1-\eta)}\frac{Y_t}{N_t} - \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\frac{Y_t}{N_t}\right) \\ &= \frac{(1-\sigma\eta)}{\sigma(1-\eta)}\end{aligned}$$

C.6.9 Prices

From A.45, $(q_{it})^{-\sigma} = N_t^{-\frac{\sigma}{\sigma-1}}$. Then:

$$\begin{aligned} q_{it} &= N_t^{\frac{1}{\sigma-1}} \\ &= (L_t \psi_c)^{\frac{1}{\sigma-1}} \end{aligned}$$

Next, from equation (53) in the paper, p_{it} is defined as $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st}$. Now, from the computation of C in C.5.5, we have $p_{it} = N_t^{\frac{1}{\sigma-1}} \left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right]$. Therefore:

$$\begin{aligned} p_{it} &= N_t^{\frac{1}{\sigma-1}} \left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] \\ &= (L_t \psi_c)^{\frac{1}{\sigma-1}} \left[1 + \frac{\eta}{1-\eta}\frac{\sigma-1}{\sigma}\right] \end{aligned}$$

C.6.10 Price p^b_{st}

Recall from A.53 that $p_{bt} = (1-\alpha)N_t^{\frac{1}{\sigma-1}} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{(\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t)}$ and we know that, by the zero profit condition in the data intermediary, $p_{bt} = \frac{p^b_{st}}{N_t}$. Then:

$$
\begin{aligned}
p^b_{st} &= p_{bt}N_t \\
&= (1-\alpha)N_t^{\frac{1}{\sigma-1}+1} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{(\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t)} \\
&= (1-\alpha)N_t^{\frac{1}{\sigma-1}+1} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{N_t\left(\frac{\alpha x_{it}}{N_t}+(1-\alpha)\tilde{x}_{it}\right)} \\
&= (1-\alpha)N_t^{\frac{1}{\sigma-1}} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{(1-\alpha)\tilde{x}_{it}} \\
&= N_t^{\frac{1}{\sigma-1}} \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{\tilde{x}_{it}} \\
&= \frac{\eta}{1-\eta} \frac{\sigma-1}{\sigma} \frac{1}{\tilde{x}_c} (L_t\psi_c)^{\frac{1}{\sigma-1}}
\end{aligned}
$$

D Competitive Equilibrium With Outlaw Data Sharing

D.1 Household Problem

The problem is

$$
\begin{aligned}
U_0 =& \max_{\{c_{it}\}} \int_0^\infty e^{-(\tilde{\rho})} L_0 u(c_t, x_{it}) dt \\
s.t.\ c_t =& \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\
\dot{a}_t =& (r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it}c_{it} di
\end{aligned}
$$

Define the current value Hamiltonian with state variable a_t, control variable c_{it}, and co-state variable μ_t:

$$
H(a_t, c_{it}, \mu_t) = u(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it}) + \mu_t \left[(r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it}c_{it} di\right]
$$

The FOCs are:

$$\begin{cases} \frac{\partial H}{\partial c_{it}} = 0 \\ \frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t \end{cases}$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma-1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\sigma-1} c_{it} = (\mu_t p_{it})^{-\sigma}$$

$$\Rightarrow \; \left(c_t^{\sigma-1} c_{it}\right)^{\frac{\sigma-1}{\sigma}} = (\mu_t p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{\sigma-1}{\sigma} + \frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; \mu_t = \frac{1}{c_t \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}}$$

Next, use the fact that the price of c_t is normalized to 1, i.e., $P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} = 1$.

Next, plug the expression for μ_t in the FOC and this gives equation (39):

$$c_{it} = c_t p_{it}^{-\sigma}$$

Next, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t(r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} + g_c$$

D.2 Firm Problem

D.2.1 Define the Problem of the Firm

The firm problem is:

$$\begin{aligned} r_t V_{it} &= \underset{\{L_{it}, D_{bit}\}}{max} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} + \dot{V}_{it} \\ s.t. \ Y_{it} &= D_{it}^{\eta} L_{it} \\ D_{it} &= \alpha x_{it} Y_{it} \\ x_{it} &\in [0; \bar{x}] \end{aligned}$$

D.2.2 Compute FOC

To solve this problem, write the Lagrangean:

$$\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} + \dot{V}_{it}. + \mu_{x0}(x_{it}) + \mu_{x1}(\bar{x} - x_{it})$$

Now take the FOCs. Start with the FOC w/r to L_{it}:

$$\begin{aligned} \frac{\partial \mathbb{L}}{\partial L_{it}} &= 0 \\ \Leftrightarrow \ (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial L_{it}} &= w_t \end{aligned}$$

And using $Y_{it} = D_{it}^{\eta} L_{it}$ and assuming D_{it} depends on L_{it}, then by implicit derivation:

$$\frac{\partial Y_{it}}{\partial L_{it}} = \eta D_{it}^{\eta-1} L_{it} \frac{\partial D_{it}}{\partial L_{it}} + D_{it}^{\eta}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

Next, compute $\frac{\partial D_{it}}{\partial L_{it}}$ using $D_{it} = \alpha x_{it} Y_{it}$:

$$\frac{\partial D_{it}}{\partial L_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}}$$

Substituting above:

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}}$$

The FOC for L_{it} is then:

$$\frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = w_t \tag{A.57}$$

Next, compute the FOC w/r to x_{it}:

$$\frac{\partial \mathbb{L}}{\partial x_{it}} = 0$$

$$\Leftrightarrow (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial x_{it}} + \mu_{x0} - \mu_{x1} = 0$$

$$\frac{\partial Y_{it}}{\partial x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = -\mu_{x0} + \mu_{x1}$$

Now to compute $\frac{\partial Y_{it}}{\partial x_{it}}$ use $Y_{it} = D_{it}^{\eta} L_{it}$:

$$\frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial x_{it}}$$

and using $D_{it} = \alpha x_{it} Y_{it}$ and implicit derivation:

$$\frac{\partial D_{it}}{\partial x_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it}\alpha$$

Thus:

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \left[\alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it}\alpha \right]$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}}$$

Then the FOC w/r to x_{it} is:

$$\frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = -\mu_{x0} + \mu_{x1}$$

Now, note that the LHS is > 0, then:

$$\mu_{x1} > \mu_{x0} \geq 0$$

$$\Rightarrow \mu_{x1} > 0$$

$$\Rightarrow x_{it} = \bar{x} \tag{A.58}$$

and the FOC w/r to x_{it} is:

$$\frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = \mu_{x1} \tag{A.59}$$

D.3 Free Entry and the Creation of New Varieties

The free entry condition is given by:

$$\chi w_t = V_{it}$$

D.4 Equilibrium with Outlaw Data Sharing

D.4.1 Expressions for Aggregate Output and Firm Output

Now, we need to solve Y_t. For that, from above $\frac{Y_{it}}{L_{it}} = N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}}$ and $L_{it} = \frac{L_{pt}}{N_t}$. Thus:

$$Y_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it} \tag{A.60}$$

But from the firm's problem we have

$$Y_{it} = D_{it}^{\eta} L_{it} \tag{A.61}$$

and $D_{it} = \alpha \bar{x} Y_{it}$. Hence:

$$\begin{aligned} Y_{it} &= (\alpha \bar{x} Y_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha \bar{x})^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha \bar{x})^{\eta} \left(\frac{L_{pt}}{N_t} \right) \end{aligned}$$

$$\Rightarrow Y_{it} = (\alpha \bar{x})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t} \right)^{\frac{1}{1-\eta}} \tag{A.62}$$

Substitute this expression for Y_{it} in A.60:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}} (\alpha \bar{x})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t} \right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha \bar{x})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \end{aligned} \tag{A.63}$$

D.4.2 Expression for w_t

From A.57, $\frac{\frac{Y_{it}}{L_{it}}}{1-\eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = w_t$, hence:

$$w_t = \frac{\frac{Y_{it}}{L_{it}}}{1-\eta \frac{Y_{it}}{D_{it}}\alpha\bar{x}}(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$$

$$= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}L_{it}^{-1}}{1-\eta\frac{Y_{it}}{\alpha\bar{x}Y_{it}}\alpha\bar{x}}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}}$$

$$= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}L_{it}^{-1}}{1-\eta}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}}$$

$$= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}N_t L_{pt}^{-1}}{1-\eta}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}}$$

$$= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}}$$

Finally:

$$w_t = \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \tag{A.64}$$

D.4.3 Expressions for Profits and Value of the Firm

The firm's problem is:

$$r_t V_{it} = \underset{\{L_{it}, D_{bit}, x_{it}, \tilde{x}_{it}\}}{max} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} + \dot{V}_{it}$$

Thus:

$$V_{it} = \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it}}{r - \frac{\dot{V}_{it}}{V_{it}}}$$

Next, define $\pi_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it}$. Then using the expressions above:

$$\begin{aligned}\pi_{it} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it} \\ &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w\left(\frac{L_{pt}}{N_t}\right) \\ &= \left(N_t^{\frac{1}{\sigma-1}}\right) N_t^{-\frac{\sigma}{\sigma-1}} Y_t - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right) \\ &= N_t^{-1}Y_t - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \\ &= N_t^{-1}N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \\ &= L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]\end{aligned} \tag{A.65}$$

Then, the value of the firm is given by:

$$\begin{aligned}V_{it} &= \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it}}{r - \frac{\dot{V}_{it}}{V_{it}}} \\ &= \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{r-g_V}\end{aligned} \tag{A.66}$$

D.5 Solution of the Competitive Equilibrium

D.5.1 Firm Size v_{os}

Use the free entry condition:

$$\chi w_t = V_{it}$$

Subsisting the expressions for w_t and V_{it} using A.64 and A.66:

$$\chi\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} = \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{r-g_V}$$

$$\chi\frac{1}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{-1}N_t = \frac{\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{r-g_V}$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi \frac{1}{1-\eta}(1-\frac{1}{\sigma})\left(r-g_V\right)}{1-\frac{\sigma-1}{\sigma(1-\eta)}}$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi \frac{1}{1-\eta}(\frac{\sigma-1}{\sigma})\left(r-g_V\right)}{\frac{\sigma(1-\eta)-(\sigma-1)}{\sigma(1-\eta)}}$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi\left(\sigma-1\right)\left(r-g_V\right)}{\sigma(1-\eta)-(\sigma-1)}$$

but $\rho = r - g_V$ then:

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi\rho\left(\sigma-1\right)}{1-\sigma\eta}$$

Define $v_{os} = \frac{L_{pt}}{N_t}$. Then:

$$v_{os} = \frac{\chi\rho\left(\sigma-1\right)}{1-\sigma\eta} \tag{A.67}$$

which is equation (76).

D.5.2 Number of varieties

From the evolution of the number of varieties, we have $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

In BGP: $\frac{\dot{N}_t}{N_t} = g_N$ and since $g_N = g_L$, then:

$$\Rightarrow g_L = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

$$\Rightarrow \frac{L_t}{N_t} = \frac{L_{pt}}{N_t} + \chi g_L$$

Next, using $\frac{\dot{V}_{it}}{V_{it}} = g_\pi$ and substituting the firm size:

$$\Rightarrow \frac{L_t}{N_t} = \frac{\chi\rho\left(\sigma-1\right)}{1-\sigma\eta} + \chi g_L$$

$$\Rightarrow N_t = \frac{L_t}{\frac{\chi\rho(\sigma-1)}{1-\sigma\eta} + \chi g_L}$$

Define $\psi_{os} = \frac{1}{\frac{\chi\rho(\sigma-1)}{1-\sigma\eta}+\chi g_L} = \frac{1}{v_{os}+\chi g_L}$ so that:

$$N_t = L_t \psi_{os} \tag{A.68}$$

which is equation (79).

D.5.3 Solution for aggregate output Y_t^{os}.

From A.63, $Y_t = N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}}$. Then: v_{os}

$$\begin{aligned} Y_t^{os} &= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}v_{os}^{\frac{1}{1-\eta}} \\ &= (\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}v_{os} \\ &= (v_{os}\alpha^{\eta}\bar{x}^{\eta})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

and this is equation (82).

D.5.4 Consumption per capita and growth

$$c_t^{os} = \frac{Y_t^{os}}{L_t} \propto L_t^{\frac{1}{\sigma-1}}$$

which is equation (84). Thus, consumption per capita growth is:

$$g_c^{os} = (\frac{1}{\sigma-1})g_L$$

which is is equation (86).

D.5.5 Firm Production Y_{it}^{os}

Using equation (82) of the paper:

$$\begin{aligned} Y_{it}^{os} &= Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\ &= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}} \left(\psi_{os}L_t\right)^{\frac{\sigma}{\sigma-1}} N_t^{-\frac{\sigma}{\sigma-1}} \\ &= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}} \left(\psi_{os}L_t\right)^{\frac{\sigma}{\sigma-1}} \left(L_t\psi_{os}\right)^{-\frac{\sigma}{\sigma-1}} \\ &= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}} \end{aligned}$$

which is equation (92).

D.5.6 Data used by the firm D_{it}^{os}

By definition, $D_{it} = \alpha x_{it} Y_{it}$. Then:

$$\begin{aligned} D_{it}^{f} &= \alpha x_{it} Y_{it} \\ &= \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}} \end{aligned}$$

which is equation (88).

D.5.7 Aggregate Data used by the firm D_t^{os}

By definition $D_t = N_t D_{it}$, hence:

$$\begin{aligned} D_{t}^{f} &= N_t \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}} \\ &= \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}} L_t \psi_{os} \end{aligned}$$

which is equation (90).

D.5.8 Labor share $\left(\frac{w_t L_{pt}}{Y_t}\right)$

From A.64:

$$\left(\frac{w_t L_{pt}}{Y_t}\right)^{os} = \frac{L_{pt}}{Y_t}\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}}$$
$$= ((v_{os}\alpha^{\eta}\bar{x}^{\eta})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}})^{-1}\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}+1}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}}$$
$$= ((v_{os})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}})^{-1}\frac{1}{1-\eta}(1-\frac{1}{\sigma})(v_{os}L_t\psi_{os})^{\frac{\eta}{1-\eta}+1}(L_t\psi_{os})^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}}$$
$$= \frac{1}{1-\eta}(1-\frac{1}{\sigma})$$
$$= \frac{1-\sigma}{\sigma(1-\eta)}$$

which is equation (93).

D.5.9 Profit share $\left(\frac{N_t\pi_t}{Y_t}\right)$

From A.65, $\pi_t = L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]$. Hence:

$$\left(\frac{\pi_t N_t}{Y_t}\right)^{os} = \frac{N_t L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{Y_t}$$
$$= \frac{L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{(v_{os}\alpha^{\eta}\bar{x}^{\eta})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}}$$
$$= \frac{(v_{os}L_t\psi_{os})^{\frac{1}{1-\eta}}(L_t\psi_{os})^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{(v_{os}\alpha^{\eta}\bar{x}^{\eta})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}}$$
$$= 1-\frac{\sigma-1}{\sigma(1-\eta)}$$
$$= \frac{1-\eta\sigma}{\sigma(1-\eta)}$$

and this is equation (94).

D.5.10 Price of a variety p_{it}^{os}

From the household problem, $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$, then:

$$\begin{aligned} p_{it}^{os} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \\ &= \left(\frac{Y_t}{Y_t N_t^{-\frac{\sigma}{\sigma-1}}}\right)^{\frac{1}{\sigma}} \\ &= N_t^{\frac{1}{\sigma-1}} \end{aligned}$$

and from equation (79), $N_t = \psi_{os} L_t$, thus:

$$p_{it}^{f} = (\psi_{os} L_t)^{\frac{1}{\sigma-1}}$$

which is equation (99).